BALANCING AND COBALANCING NUMBERS

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CERTIFICATE

This is to certify that the thesis entitled “Balancing and Cobalancing Numbers” which is being submitted by Mr. Prasanta Kumar Ray, Roll No. 50612001, for the award of the degree of Doctor of Philosophy from National Institute of Technology, Rourkela, is a record of bonafide research work, carried out by him under my supervision. The results embodied in this thesis are new and have not been submitted to any other university or institution for the award of any degree or diploma.

To the best of my knowledge, Mr. Ray bears a good moral character and is mentally and physically fit to get the degree.

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(Prasanta Kumar Ray)

ABSTRACT

A different approach to the theory of balancing numbers is possible by means of a Pell's equation which can be derived from the definition of balancing numbers. Each balancing number corresponds to a balancer, and each balancer, in turn, is a cobalancing number. Similarly, each cobalancing number corresponds to a cobalancer and interestingly, each cobalancer is a balancing number. The cobalancing numbers as well as their cobalancers are also solutions of a Diophantine equation similar to that satisfied by balancing numbers and their balancers. Some Diophantine equations exhibit beautiful solutions in terms of balancing and cobalancing numbers. The Lucas-balancing and Lucas-cobalancing numbers, obtained respectively as functions of balancing and cobalancing numbers, are useful in the computation of balancing and cobalancing numbers of higher order. Pell and associated Pell numbers are very closely associated with balancing and cobalancing numbers and appear in the factorization and as greatest common divisors of these numbers. The balancing, cobalancing and other related numbers are also expressible in terms of products of matrices.

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CHAPTER 0

Introduction

There is a quote by the famous mathematician Carl Friedrich Gauss (1777– 1855): "Mathematics is the queen of all sciences, and number theory is the queen of mathematics." Number theory, or higher arithmetic is the study of those properties of integers and rational numbers, which go beyond the ordinary manipulations of everyday arithmetic.

Throughout history, almost every major civilization has been fascinated by the properties of integers and has produced number theorists. In ancient and medieval times, these were usually geometers, or more generally scholars, calendar calculators, astronomers, astrologers, priests or magicians.

The oldest number theoretical record we have is a tablet from Babylonia, a table of right triangles with integer sides, that is, positive integer solutions to $x^2 + y^2 = z^2$. Some of these solutions are too large for us to believe that they were discovered by trial and error in those days. The Babylonian scholars knew the Pythagorean Theorem well over a millennium before Pythagoras and were also able to compute with large numbers.

Euclid and Diophantus of Alexandria (about 300-200 B.C.) are the best known number theorists of ancient times. Euclid's contribution consists of thirteen books, three of them are about number theory of the positive integers, but everything is stated in a geometric language. Among these results, Euclid's contributions are the properties of divisibility of numbers including the idea of odd and even numbers, and an algorithm for finding the greatest common divisor of two numbers. He derived formulas for the sum of a finite geometric progression and for all Pythagorean triples. He introduced the notion of a prime number and showed that if a prime number divides a product of two numbers, it must divide at least one of them. He also proved the infinitude of primes in the same way we are doing till today.

Diophantine analysis was named after Diophantus, who proposed many indeterminate problems in his book Arithmetica. For example, he desired three rational numbers, the product of any two of which increased by the third shall be a square. Again, he wanted that certain combinations of the sides, area and perimeter of a right triangle shall be squares or cubes. He was satisfied with a single numerical rational solution, in spite of his problems usually having infinitude of such solutions. Although Diophantus is primarily interested in solving equations in positive rational numbers, equations with solutions to be found in integers are now referred to as Diophantine equations.

One of the most famous mathematical problems of all time in Diophantine analysis is Fermat's last theorem [15, 17, 19, 43], which states that the Diophantine equation $x^n + y^n = z^n$ has no solution in positive integers x, y and z if $n \geq 3$. Pierre de Fermat (1601– 1665) was a judge in Toulouse, France and also a very serious amateur mathematician. One evening, reading a copy of Diophantus' Arithmetica, newly rediscovered and translated from Greek to Latin, he came on a theorem about Pythagorean triplets. In the margin of the book he wrote “It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or, in general, any power higher than the second into two like powers. I have discovered a truly marvelous proof of this result, which, this margin is too narrow to contain.” Fermat left no proof of the conjecture for all n, but he did prove the special case $n = 4$. This reduced the problem to proving the theorem for all exponents n that are odd prime numbers. Over the next two centuries (1637– 1839), the conjecture was proved for only the first three odd primes (3, 5, and 7), although Sophie Germain (1776 − 1831) proved a special case for all primes less than 100 [48, p. 109]. In the mid-19^{th} century, Ernst Kummer (1810 – 1893) proved the theorem for a large (probably infinite) class of primes known as regular [16, p.228]. Building on Kummer's work and using sophisticated computer studies, other mathematicians were able to prove the conjecture for all primes up to four million. Despite much progress in special cases, the problem remained unsolved until Andrew Wiles, a British mathematician, working at Princeton University, announced his solution in 1992 and corrected in 1995 [25]. Wiles’ proof, not only settled an old mathematical problem, but it also opened the doors to new areas of research thought with the introduction of new ideas and techniques.

The notion of figurate numbers, and in particular, triangular numbers goes back to Pythagoras, who represented them by points arranged as are the shot in the base of triangular pile of shot. The number of shot in such a pile is called a tetrahedral number. In an analogous manner, one can define a polygonal number of m sides and a pyramidal number. The most important result on this subject first stated by Fermat is: "Every positive integer is either triangular or a sum of two or three triangular numbers; every positive integer is either a square or a sum of two, three or four squares; either pentagonal or a sum of 2, 3, 4 or 5 pentagonal numbers and so on.”

There is an interesting story with triangular numbers while Gauss was a school student. The teacher asked everyone in the class to find the sum of all the numbers from 1 to 100. To everybody's surprise, Gauss stood up with the answer 5050 immediately. The teacher asked him as to how it was done. Gauss explained that instead of adding all the numbers from 1 to 100, add first and last term i.e., $1 + 100 = 101$, then add second and second last term i.e., $2 + 99 = 101$ and so on. The sum of every pair is 101 and there will be 50 such pairs, so $101 \times 50 = 5050$ is the answer. Hence, the sum of numbers from 1 to n is $(n/2) \times (n + 1)$, where $n/2$ are the number of pairs and $n + 1$ is sum of each pair. This is the famous formula for the n^{th} triangular number.

In number theory, discovery of number sequences with certain specified properties has been a source of attraction since ancient times. The most beautiful and simplest of all number sequences is the Fibonacci sequence [18, 28, 36, 61]. This sequence was first invented by Leonardo of Pisa $(1180 - 1250)$, who was also known as Fibonacci, to describe the growth of a rabbit population. It describes the number of pairs in a rabbit population after n months if it is assumed that

- the first month there is just one newly born pair,
- newly born pairs become productive from their second month on,
- there is no genetic problems whatsoever generated by inbreeding,
- each month every productive pair begets a new pair, and
- the rabbits never die.

Thus, if in the n^{th} month, we have a rabbits and in the $(n+1)^{st}$ month, we have b rabbits, then in the $(n+2)^{nd}$ month we will necessarily have $a + b$ rabbits. That's because we know each rabbit basically gives birth to another each month (actually each pair gives birth to another pair, but it's the same thing) and that means that all a rabbits give birth to another number of a rabbits, become fertile after two months, which is exactly in the $(n+2)^{nd}$ month. That's why we have the population at moment $n + 1$ (which is b) plus exactly the population at moment n (which is a).

Perhaps the greatest investigator of the properties of the Fibonacci and related number sequences was François Edouard Anatole Lucas ($1842 - 1891$). A sequence related to the Fibonacci sequence bears his name, called the Lucas sequence [28], in which the first term is 1, second term is 3 and satisfy recurrence relation identical to that of Fibonacci numbers. The number of ways of picking a set (including the empty set) from the cyclic set $\{1, 2, \cdots, n\}$ without picking two consecutive numbers is given by the n^{th} Lucas number [29, p.122].

Other interesting number sequences are the Pell sequence and the associated Pell sequence [22]. In mathematics, the Pell numbers are infinite sequence of integers that have been known since ancient times, the denominators of the closest rational approximations to the square root of 2. This sequence of approximations begins with $1/1, 3/2, 7/5, 17/12$ and $41/29$; so the sequence of Pell numbers begins with $1, 2, 5, 12$ and 29. The numerators of the same sequence of approximations give the associated Pell sequence.

The Pell as well as the associated Pell sequence may be calculated by means of a recurrence relation similar to that for the Fibonacci sequence, and both sequences of numbers grow exponentially, proportionally to powers of the silver ratio $1 + \sqrt{2}$. As well as being used to approximate the square root of two, Pell and associated Pell numbers can be used to construct square triangular numbers and nearly isosceles integer right triangles, and to solve certain combinatorial enumeration problems.

It is important to note that, many of the known number sequences are solutions of Diophantine equations. A special type of Diophantine equation is the Pell's equation $x^2 - dy^2 = \pm 1$ where d is a natural number which is not a perfect square [3, 6, 33]. Indeed, the English mathematician John Pell (1610 –1685) has nothing to do with this equation. Euler (1707– 1783) mistakenly attributed to Pell, a solution method that had, in fact, been found by another English mathematician William Brouncker (1620–1684) in response to a challenge by Fermat (1601–1665); but attempts to change the terminology introduced by Euler have always proved futile. Pell's equation has an extraordinarily rich history, to which Weil's book [62] is the best guide. Brouncker's method [17, p. 351] is, in substance, identical to a method that was known to Indian mathematicians at least six centuries earlier. The equation also occurred in Greek mathematics, but no convincing evidence that the Greeks could solve the equation has ever emerged. A particularly lucid exposition of the Indian or English method of solving the Pell's equation is found in Euler's Algebra [21, p. 352]. Modern textbooks [e.g. 42] usually give a formulation in terms of continued fractions, which is also due to Euler. Euler, as well as his Indian and English predecessors, appears to take it for granted that the method always produces a solution. That is true, but it is not obvious; all that is obvious is that, if there is a solution, the method will find one. Fermat was probably in possession of a proof that there is a solution for every d [62, p.92], and Lagrange was first to publish such a proof in the year 1768.

A very recently discovered number sequence is the sequence of balancing numbers [7] by A. Behera and G. K. Panda. They call a natural number n, a balancing number if, $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$ for some natural number r, while they call r, the balancer corresponding to the balancing number n. Since the topic is new, Chapter 2 is entirely devoted to the study of balancing numbers. The balancing numbers, though obtained from a simple Diophantine equation, are very useful for the computation of square triangular numbers. An important result about balancing numbers is that, n is a balancing number if and only if $8n^2 + 1$ is a perfect square, and the number $\sqrt{8n^2 + 1}$ is called a Lucas-balancing number. The most interesting fact about Lucas-balancing numbers is that, these numbers are associated with balancing numbers in the way Lucas numbers are associated with Fibonacci numbers. The Fibonacci numbers satisfy the identity

$\left[\frac{L_n+\sqrt{5}F_n}{2}\right]^k = \frac{L_{nk}+\sqrt{5}F_{nk}}{2}$, where F_n is the n^{th} Fibonacci number and L_n is the n^{th} Lucas number. In the same way, the balancing numbers satisfy $\left(C_n+\sqrt{8}B_n\right)^k = C_{nk}+\sqrt{8}B_{nk}$ [see 44], where B_n is the n^{th} balancing number and C_n is the n^{th} Lucas-balancing number. Both the results resemble the De-Moivre's theorem $(\cos x + i\sin x)^k = \cos kx + i\sin kx$ of complex analysis [1]. The Fibonacci numbers satisfy $F_{m+n} = \frac{F_mL_n+L_mF_n}{2}$, while Panda [44] proved that the balancing numbers satisfy $B_{m+n} = B_mC_n + C_mB_n$, which looks like the trigonometric identity $\sin(x+y) = \sin x\cos y + \cos x\sin y$. We observe that, the identity for balancing numbers looks more symmetric than that for Fibonacci numbers. There are some other properties common to both Fibonacci as well as balancing numbers. For example, like Fibonacci numbers, the greatest common divisor of any two balancing number is a balancing number. Further, m divides n, if and only if the m^{th} balancing number divides the n^{th} balancing number. More generally, the greatest common divisor of the m^{th} and n^{th} order balancing numbers is equal to the balancing number whose order is equal to the greatest common divisor of m and n. The balancing numbers, behave in many cases, like the natural numbers. Just like the sum of first n odd natural numbers is equal to n^2, Panda [44] showed that the sum of first n odd balancing numbers is equal to the square of the n^{th} balancing number. This property is not true for the Fibonacci numbers.

As shown by Behera and Panda in [7], the square of a balancing number is a triangular number, and indeed, all square triangular numbers can be generated in this way. Now question arises, whether the square of a triangular number can be another triangular number. Though Luo [37] answered to this question affirmative, one shouldn't be much enthusiastic about this answer, because what Luo proved is that that the only triangular numbers whose squares are also triangular are 1 and 6. Indeed, Luo's answer solves an important question concerning balancing numbers that the only balancing numbers which are also triangular are 1 and 6.

Panda [45] generalized balancing and cobalancing numbers by introducing sequence balancing and cobalancing numbers, in which, the sequence of natural numbers, used in the definition of balancing and cobalancing numbers is replaced by an arbitrary sequence of real numbers. Thus if $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, then a_k is called a sequence balancing number if $a_1 + a_2 + \cdots + a_{k-1} = a_{k+1} + a_{k+2} + \cdots + a_{k+r}$ for some natural number r; a_k is called a sequence cobalancing number if $a_1 + a_2 + \cdots + a_k = a_{k+1} + a_{k+2} + \cdots + a_{k+r}$ for some natural number r. It has been proved in [45] that, there no sequence balancing number in the Fibonacci sequence, and the only sequence cobalancing number in this sequence is 1. He further showed that sequence balancing numbers in odd natural numbers are the sums of two consecutive balancing numbers, and there doesn't exist any sequence cobalancing number in this sequence. He also introduced the concept of higher order balancing and cobalancing numbers taking $a_n = n^k$, where k is a natural number. For $k = 1$, the higher order balancing and cobalancing numbers are nothing but the balancing and cobalancing numbers respectively. For $k = 2$ and $k = 3$, Panda [45] call the higher order balancing numbers, the balancing squares and balancing cubes; similarly, for higher order cobalancing numbers, cobalancing squares and cobalancing cubes respectively. In [45] he proved the nonexistence of balancing cubes and cobalancing cubes. For other higher order balancing and cobalancing numbers, he concluded his work with a conjecture, which states that "There exist no higher order balancing or cobalancing numbers if $k \geq 2$."

The work of Panda [45] on higher order cobalancing numbers is related to some classical unsolved problems in Diophantine analysis. In this context, there are some important works of Berstein [8, 9, 10] which, in turn, are particular cases of a problem due to Erdös [20], namely whether the Diophantine equation $m(m+1)(m+2)\cdots(m+k-1) = 2n(n+1)(n+2)\cdots(n+k-1)$ has any solution for $k > 2$ and $m + k + 1 < n$. Makowski [38] answered Erdös' question in the negative for a particular case using the results of Segal [49]. The definition of cobalancing squares is equivalent to $m(m+1)(m+2) = 2n(n+1)(n+2)$, which is a particular case of the Diophantine equation stated earlier. Mordell [40] looked at particular cases of nearly pyramidal numbers (i.e. any number differing from a pyramidal number by 1) as did Boyd and Kisilevsky [12], but the scope of generalization is wide open.

The search for balancing numbers in well known integer sequences was first initiated by Liptai [35]. He proved that there is no balancing number in the Fibonacci sequence other than 1. Subsequently, while dealing with the resolution of simultaneous Pell equations, Szalay [55] proved that there is no balancing number in the Lucas sequence other than 1.

The present work has been divided into six chapters. In Chapter 1, definition of some number sequences, their Binet formulas and other useful techniques are presented. Indeed, the historical notes to some of these sequences are also available in this chapter.

As mentioned earlier, Chapter 2 is entirely devoted to the study of balancing numbers. Though, the entire content of Chapter 2 is not a part of this work, it is included in this thesis to make it self-contained. Indeed, our approach to balancing numbers and methods of proving the results are different in many cases, and some new results are also included.

A number sequence very closely associated with the balancing number sequence is the cobalancing number sequence. A cobalancing number is a natural number n satisfying the Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$ for some natural number r. The number r is called the cobalancer corresponding to the cobalancing number n. Just like balancing numbers are related to square triangular numbers, the cobalancing numbers are linked to pronic triangular numbers (triangular numbers that are expressible as product of two consecutive natural numbers or, approximately as arithmetic mean of squares of two consecutive natural numbers). Chapter 3 is devoted to the study of cobalancing numbers and cobalancers.

It is worth mentioning here, the justification of the names cobalancing numbers and cobalancers in relation to balancing numbers and balancers respectively. In Chapter 3 it is established that every balancing number is a cobalancer, and every cobalancing number is a balancer. More precisely, the n^{th} balancing number is the $(n+1)^{st}$ cobalancer and the n^{th} cobalancing number is the n^{th} balancer. Indeed the limiting ratio of consecutive balancing numbers and consecutive cobalancing numbers, larger to smaller, both are equal to $3+2\sqrt{2}$.

The definitions and properties of balancing and cobalancing numbers indicate their close association with square and pronic triangular numbers. In [51, 52, 53], Subramaniam studied many interesting properties of square triangular numbers without realizing that the square roots of these numbers are balancing numbers. He further advanced his study with the introduction of almost square triangular numbers [54] (triangular numbers that differ from square by unity), and linked them to square triangular numbers. Subsequently, Panda [45] established the close association between almost square triangular numbers and balancing numbers.

It is interesting to note that two other important number sequences, namely, the Pell sequence and the associated Pell sequence [11] are very closely associated with the sequence of balancing and cobalancing numbers. Chapter 4 is entirely devoted to study some of such relationships. The most interesting results presented here are that, the balancing number of any order is product of the Pell and associated numbers of same order and each cobalancing number is also a product of a Pell number and an associated Pell number, though not of same order. More interestingly, if a sequence is generated from the Pell sequence dividing each term of the sequence by two, then it absorbs both the sequences of balancing and cobalancing numbers. The associated Pell sequence exhausts two sequences generated from balancing and cobalancing numbers, namely, the sequences of Lucas-balancing and the Lucas-cobalancing numbers. Pell and associated Pell numbers also appear as the greatest common divisors of two consecutive balancing numbers or cobalancing numbers or, a pair of balancing and cobalancing numbers of same order. Balancing and cobalancing numbers also arise in the partial sums of even ordered Pell numbers, odd order Pell numbers, even ordered associated Pell numbers, odd order associated Pell numbers, and in the partial sum of these numbers up to even and odd order. We also discuss the occurrence of balancing, cobalancing, Pell and associated Pell numbers in the solution of some Diophantine equations, including some Pythagorean equations. These numbers also appear in the solution of the almost Pythagorean equation, that is, the positive integral solutions to the equations $x^2 + y^2 = z^2 \pm 1$.

Some other interesting properties of balancing and related numbers are included in Chapter 5, using the limiting ratio $\lambda_1 = 3 + 2\sqrt{2}$ of two consecutive balancing numbers. Bounds to the n^{th} balancing numbers in powers of λ_1 and bounds to the n^{th} power of λ_1 in terms of balancing numbers are obtained. The Binet formula for balancing numbers is reestablished using eigen values and eigen vector techniques and finally, the four number sequences namely, balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are obtained as product of suitable matrices.

CHAPTER 1

Preliminaries

In this chapter we recall some definitions and known results on Fibonacci numbers, Lucas numbers, Pell numbers, associated Pell numbers, triangular numbers, pronic numbers, recurrence relations, Binet formula, Diophantine equations including Pell's equations and Pythagorean equations. This chapter serves as base and background for the study of subsequent chapters. We shall keep on referring back to it as and when required.

1.1 Recurrence Relation $[4, 24, 41]$. In mathematics, a *recurrence relation* is an equation that defines a sequence recursively; each term of the sequence is defined as a function of the preceding terms.

1.2 Generating Function $[32, 56]$. The (ordinary) *generating function* of a sequence $\{x_n\}_{n=1}^{\infty}$ of real or complex numbers is given by $f(s) = \sum_{n=1}^{\infty} x_n s^n$. Hence, the n^{th} term of the sequence is obtained as the coefficient of s^n in the power series expansion of $f(s)$, or by means of the formula $x_n = \frac{1}{n!}\frac{d^n}{ds^n} f(s)|_{s=0}$.

1.3 Fibonacci Sequence $[5, 23, 31, 57, 59]$. The first two *Fibonacci numbers* are $F_1 = 1, F_2 = 1$ and other terms of the sequence are obtained by means of the recurrence relation $F_{n+1} = F_n + F_{n-1}, n \geq 2$.

1.4 Lucas Sequence [31, 57]. *Lucas sequence* is also obtained from the same recurrence relation as that for Fibonacci numbers. The first two Lucas numbers are $L_1 = 1, L_2 = 3$ and other terms of the sequence are obtained by means of the recurrence relation $L_{n+1} = L_n + L_{n-1}, n \geq 2$.

1.5 Pell Sequence [22]. The first two *Pell numbers* are $P_1 = 1$, $P_2 = 2$ and other terms of the sequence are obtained by means of the recurrence relation $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 2$.

1.6 Associated Pell Sequence [22]. *Associated Pell sequence* is also obtained from the same recurrence relation as that of Pell numbers. The first two associated Pell numbers are $Q_1 = 1$, $Q_2 = 3$ and other terms of the sequence are obtained by means of the recurrence relation $Q_{n+1} = 2Q_n + Q_{n-1}, n \geq 2$.

1.7 Triangular Numbers [14, 17, 56]. A number of the form $n(n+1)/2$ where $n \in \mathbb{Z}^+$ is called a *triangular number*.

The justifications for the name triangular number are many. One such reason may be the fact that the triangular number $n(n+1)/2$ represents the area of a right angled triangle with base $n+1$ and perpendicular n.

It is well known that $m \in \mathbb{Z}^+$ is a triangular number if and only if $8m + 1$ is a perfect square.

A number which is triangular as well as a square is called a square triangular number [2, 60]. Examples of square triangular numbers are $1, 36, 1225$ and so on.

While the triangular numbers satisfy the recurrence relation $T_1 = 1$ and $T_{n+1} = T_n + (n+1)$ for $n \geq 1$ the square triangular numbers satisfy the recurrence relation $S_1 = 1$, $S_2 = 36$ and $S_{n+1} = 34S_n - S_{n-1} + 2$ for $n \geq 2$ [46, 47].

1.8 Pronic Numbers [14]. A number of the form $n(n+1)$ where $n \in \mathbb{Z}^+$ is called a *pronic number*. Some authors prefer to use the term rectangular number because each number represents the area of a rectangle with sides differing by one.

Like triangular numbers, one important result about pronic numbers is that, $m \in \mathbb{Z}^+$ is a pronic number if and only if $4m+1$ is a perfect square. But unlike triangular numbers, no pronic number can be a perfect square since consecutive natural numbers cannot be simultaneously squares. Indeed, there are numbers which are triangular and pronic at the same time. The numbers $6, 210$ and 7140 are both pronic as well as triangular.

1.9 Diophantine Equation [3, 13, 17, 41]. In mathematics, a *Diophantine equation* is an indeterminate polynomial equation that allows the variables to be integers only. Diophantine problems have fewer equations than unknowns and involve finding integers that work correctly for all the equations. In more technical language, they define an algebraic curve, algebraic surface or more general object, and ask about the lattice points on it.

1.10 Pell's Equation [6, 33, 34, 56]. The equation $x^2 - dy^2 = \pm 1$, to be solved in positive integers x and y for a given positive integer d (which is not a perfect square), is called the *Pell's equation*. For example, for $d = 5$ one can take $x = 9, y = 4$.

One may rewrite Pell's equation as $(x + dy)(x - dy) = \pm 1$, so that finding a solution comes down to finding a nontrivial unit of the ring $\mathbb{Z}[\sqrt{d}]$ of norm 1; here the norm $\mathbb{Z}[\sqrt{d}]^* \longrightarrow \mathbb{Z}^* = \{\pm 1\}$ between unit groups multiplies each unit by its conjugate, and the units ± 1 of $\mathbb{Z}[\sqrt{d}]$ are considered trivial. This reformulation implies that, knowing a single solution to Pell's equation, one can easily find infinitely many. More precisely, if the solutions are ordered in ascending order of magnitude, then the n^{th} solution $(x_n,\ y_n)$ can be expressed in terms of the first one $(x_1,\ y_1)$, by $x_n + y_n d = (x_1 + y_1 d)^n$. Accordingly, the first solution (x_1, y_1) is called the fundamental solution to the Pell equation, and solving the Pell's equation means finding (x_1, y_1) for a given d.

1.11 Pythagorean Triplets. [17, 26, 27, 58]. The equation $x^2 + y^2 = z^2$, where $x, y, z \in \mathbb{Z}^+$ is called the *Pythagorean equation* and three integers x, y and z that satisfy $x^2 + y^2 = z^2$ are called *Pythagorean triplets*.

There are infinitely many such triplets and there also exists a way to generate all the triplets. Let n, m and k be natural numbers such that $n > m$. Define

$$x = k(n^2 - m^2),\ y = 2knm,\ z = k(n^2 + m^2).$$

It is easy to see that the three numbers x, y and z always form a Pythagorean triplet.

First of all, note that if $x^2 + y^2 = z^2$, then $\left[\frac{x}{z}\right]^2 + \left[\frac{y}{z}\right]^2 = 1$. With $u = x/z$ and $v = y/z$, we get $u^2 + v^2 = 1$. This is the well known equation of the unit circle with center at the origin. Finding Pythagorean triplets is therefore equivalent to locating rational points (i.e., points (u, v) for which both u and v are rational) on the unit circle.

1.12 Binet Formula [5, 30]. While solving a recurrence relation as a difference equation, the n^{th} term of the sequence is obtained in closed form, which is a formula containing conjugate surds of irrational numbers is known as the *Binet formula* for the particular sequence. These surds are obtained from the auxiliary equation of the recurrence relation for the recursive sequence under consideration. The Binet formula for the Fibonacci sequence is $F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$, where φ is the golden ratio [18, 36, 61], equal to $\frac{1+\sqrt{5}}{2}$. The Binet formula for the Lucas sequence also depends on this golden ratio and is given by $L_n = \varphi^n + (1-\varphi)^n$. Similarly, the Binet formulas of Pell and associated Pell sequences are $P_n = (\alpha_1^n - \alpha_2^n)/2\sqrt{2}$, $Q_n = (\alpha_1^n + \alpha_2^n)/2$, where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$. The Binet forms of other sequences will be discussed in the relevant chapters.

CHAPTER 2

Balancing Numbers

2.1 INTRODUCTION

The concept of *balancing numbers* was first introduced by Behera and Panda [7] in the year 1999 in connection with a Diophantine equation. It consists of finding a natural number n such that

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \qquad \cdots (1)$$

for some natural number r. They call n a balancing number and r, the *balancer* corresponding to n. If the n^{th} triangular number $n(n+1)/2$ is denoted by T_n, then equation (1) reduces to

$$T_{n-1}+T_n=T_{n+r},$$

which is the problem of finding two consecutive triangular numbers whose sum is also a triangular number. Since

$$\begin{aligned} T_5+T_6&=15+21\\ &=36=T_8, \end{aligned}$$

6 is a balancing number with balancer 2. Similarly,

$$\begin{aligned} T_{34}+T_{35}&=595+630\\ &=1225=T_{49}, \end{aligned}$$

implies that, 35 is also a balancing number with balancer 14. But finding two consecutive triangular numbers such that their sum is also a triangular number is not an easy task. So, some easy method of finding balancing numbers is required. Since the sum of two consecutive triangular numbers is a perfect square, the search for balancing numbers reduces to finding triangular numbers that are perfect squares. Indeed, (1) simplifies to

$$\frac{(n+r)(n+r+1)}{2} = n^2, \qquad \cdots (2)$$

and if n is known, then r can be obtained in terms of n as

$$r = \frac{-(2n+1)+\sqrt{8n^2+1}}{2}. \qquad \cdots (3)$$

It follows from (2) and (3) that n is a balancing number, if and only if $8n^2 + 1$ is a perfect square. Further, if the m^{th} triangular number is a perfect square equal to n^2 then n is a balancing number with balancer $m - n$. Since $T_8 = 36 = 6^2$, 6 is a balancing number with balancer 2. Similarly, since $T_{49} = 1225 = 35^2$, 35 is a balancing number with balancer 14.

Indeed, each square triangular number is a figurate number expressible as both a square and a triangle. For example,

$$\frac{m(m+1)}{2} = n^2$$

is a square triangular number and by virtue of (2), n is a balancing number, m is equal to n plus its balancer r, and n becomes a side of the square and m, a side of the triangle. The following figure represents the square and triangle corresponding to the balancing number 6.

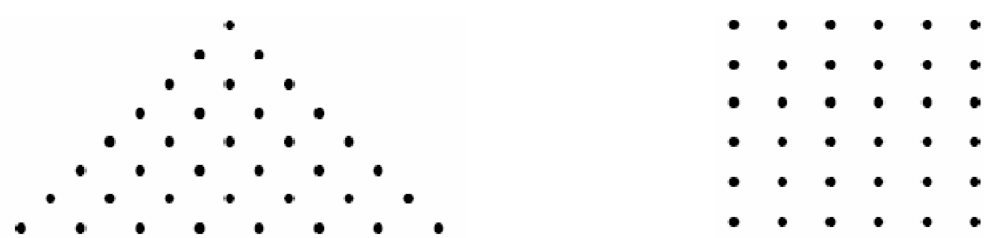

Figure 2.1: The square triangular number 36 as triangle and square, side of triangle $n + r = 8$, side of square $n = 6$

2.2 BINET FORMULA FOR BALANCING NUMBERS

In [7], Behera and Panda first obtained the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1},$$

and then developed the Binet formula by solving this recurrence relation as a second order linear homogeneous difference equation. In this section, we first obtain the Binet formula for balancing numbers using Pell's equation [33, 34] and develop the above stated recurrence relation in the next section.

It is seen in Section 2.1 that x is a balancing number if and only if $8x^2 + 1$ is a perfect square. Writing

$$8x^2 + 1 = y^2,$$

our search for balancing numbers reduces to solving the equation

$$y^2 - 8x^2 = 1, \qquad \cdots (4)$$

which is the Pell's equation with $d = 8$. The fundamental solution of (4) is $y_1 = 3$ and $x_1 = 1$. Hence the totality of solutions can be obtained from

$$y_n + \sqrt{8}x_n = \left(3 + \sqrt{8}\right)^n. \qquad \cdots (5)$$

Since (5) implies

$$y_n - \sqrt{8}x_n = \left(3 - \sqrt{8}\right)^n, \qquad \cdots (6)$$

subtracting (6) from (5) and dividing the resultant equation by $2\sqrt{8}$, we get

$$x_n = \frac{\left(3+\sqrt{8}\right)^n - \left(3-\sqrt{8}\right)^n}{2\sqrt{8}}. \qquad \cdots (7)$$

The n^{th} balancing number is denoted by B_n [7, 44], and hence, the right hand side of (7) gives the Binet formula for B_n. In [7], Behera and Panda call 6 the first balancing number since 6 is the smallest natural number satisfying (1). As $8 \cdot 1^2 + 1$ is a perfect square, and for $n = 1$, the right hand side of (7) is equal to 1, Panda [44] call 1 as the first balancing number, that is, $B_1 = 1, B_2 = 6$ and so on. Taking $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$, the Binet formula for B_n can be written as

$$B_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}}. \qquad \cdots (8)$$

2.3 RECURRENCE RELATIONS FOR BALANCING NUMBERS

In this section, we first obtain the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}$$

using the Binet formula (8) and then present some other recurrence relations for balancing numbers obtained by Behera and Panda [7] and Panda [44], though the methods of derivations are different in many cases.

Using the Binet formula (8), and observing that $\alpha_1\alpha_2 = -1$, we obtain

$$\begin{aligned} B_{n+1} + B_{n-1} &= \frac{\alpha_1^{2(n+1)} - \alpha_2^{2(n+1)}}{4\sqrt{2}} + \frac{\alpha_1^{2(n-1)} - \alpha_2^{2(n-1)}}{4\sqrt{2}} \\ &= \frac{\alpha_1^{2n}(\alpha_1^2 + \alpha_2^2) - \alpha_2^{2n}(\alpha_1^2 + \alpha_2^2)}{4\sqrt{2}} \\ &= 6\,\frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \\ &= 6B_n \quad (n \geq 2). \end{aligned}$$

This gives,

$$B_{n+1} = 6B_n - B_{n-1},\ n \geq 2, \qquad \cdots(9)$$

which is the required recurrence relation for balancing numbers. This above discussion proves the following theorem.

2.3.1 Theorem. *The sequence* $\{B_n\}_{n=1}^{\infty}$ *of balancing numbers satisfies the linear recurrence relation* $B_1 = 1, B_2 = 6,$ *and* $B_{n+1} = 6B_n - B_{n-1}, n \geq 2.$

The following theorem gives a nonlinear recurrence relation for balancing numbers.

2.3.2 Theorem [7]**.** *The balancing numbers satisfy the nonlinear second order recurrence relation* $B_n^2 = 1 + B_{n-1}B_{n+1}, n \geq 2.$

Proof. The proof is based on induction. The case $n = 2$ is obvious. Assuming that

$$B_n^2 = 1 + B_{n-1}B_{n+1}$$

is true for $n = k$, and using Theorem 2.3.1, we get

$$\begin{aligned} B_{k+1}^2 &= (6B_k - B_{k-1})B_{k+1} \\ &= 6B_kB_{k+1} - B_{k-1}B_{k+1} \\ &= 6B_kB_{k+1} - (B_k^2 - 1) \\ &= B_k(6B_{k+1} - B_k) + 1 \\ &= B_kB_{k+2} + 1, \end{aligned}$$

showing that the relation is true for $n = k + 1$. ■

2.3.3 Remark. The rough idea in Theorem 2.3.2 is that, any three consecutive balancing numbers are approximately in geometric progression. In particular

$$B_n = \left\lceil \sqrt{B_{n-1}B_{n+1}} \right\rceil,$$

where $\lceil\cdot\rceil$ denotes the ceiling function which maps a real number to the smallest integer greater than or equal to it, or more precisely, for any real number x, $\lceil x \rceil = \lfloor x + 1 \rfloor$.

There are some other interesting relationships among balancing numbers. Using the recurrence relation (9), the following theorem establishes one such relationship. Indeed, we need $B_0 = 0$, which can be obtained easily from the Binet formula (8).

2.3.4 Theorem. *For any two positive integers m and n, $B_{m+n} = B_mB_{n+1} - B_{m-1}B_n$.*

Proof. The theorem is proved by induction. The recurrence relation (9) shows that the assertion is true for $n = 1$. Assuming the assertion true for $n \leq k$, we have

$$\begin{aligned} B_{m+k+1} &= 6B_{m+k} - B_{m+k-1} \\ &= 6(B_mB_{k+1} - B_{m-1}B_k) - (B_mB_k - B_{m-1}B_{k-1}) \\ &= B_m(6B_{k+1} - B_k) - B_{m-1}(6B_k - B_{k-1}) \\ &= B_mB_{k+2} - B_{m-1}B_{k+1}, \end{aligned}$$

showing that the assertion is also true for $n = k + 1$. ■

In the last theorem, substituting $m = n - 1$ and $m = n$ respectively, and using Theorem 2.3.2, we get the following results.

2.3.5 Corollary. *For any positive integer n, the balancing numbers satisfy*

(a) $B_{2n-1} = B_n^2 - B_{n-1}^2$,

(b) $B_{2n} = B_n(B_{n+1} - B_{n-1})$.

The following corollary is an easy consequence of Corollary 2.3.5.

2.3.6 Corollary. *If n is a natural number, then*

(a) $B_1 + B_3 + \cdots + B_{2n-1} = B_n^2$,

(b) $B_2 + B_4 + \cdots + B_{2n} = B_n B_{n+1}$.

Now, we are in a position to find the recurrence relation for square triangular numbers using Theorems 2.3.1 and 2.3.2.

2.3.7 Theorem. *The sequence of square triangular numbers $\{ST_n\}_{n=1}^{\infty}$ satisfies the recurrence relation $ST_1 = 1$, $ST_2 = 6$ and $ST_{n+1} = 34ST_n - ST_{n-1} + 2$ for $n \geq 3$.*

Proof. Squaring both sides of the recurrence relation (9), we get

$$\begin{aligned} B_{n+1}^2 &= 34B_n^2 - (12B_nB_{n-1} - 2B_n^2 - B_{n-1}^2) \\ &= 34B_n^2 - 2(B_{n+1} + B_{n-1})B_{n-1} + 2B_n^2 + B_{n-1}^2 \\ &= 34B_n^2 + 2(B_n^2 - B_{n+1}B_{n-1}) - B_{n-1}^2. \end{aligned}$$

By virtue of Theorem 2.3.2,

$$B_n^2 - B_{n+1}B_{n-1} = 1,$$

and we have

$$B_{n+1}^2 = 34B_n^2 - B_{n-1}^2 + 2.$$

Since

$$ST_n = B_n^2,$$

it follows that

$$ST_{n+1} = 34ST_n - ST_{n-1} + 2.$$

This completes the proof. ■

2.4 FUNCTIONS GENERATING BALANCING NUMBERS

In this section, we present some functions discussed in [7] generating balancing numbers. Given any arbitrary balancing number x, we consider the following functions:

$$f(x) = 2x\sqrt{8x^2+1},$$

$$g(x) = 3x + \sqrt{8x^2+1},$$

$$h(x) = 17x + 6\sqrt{8x^2+1},$$

$$p(x) = 6x\sqrt{8x^2+1} + 16x^2 + 1.$$

The following theorem shows that the above functions always generate balancing numbers.

2.4.1 Theorem [7]. *If x is a balancing number, then the functions $f(x), g(x), h(x)$ and $p(x)$ are also balancing numbers.*

Proof. Since x is a balancing number, $8x^2+1$ is a perfect square and

$$\frac{8x^2(8x^2+1)}{2} = 4x^2(8x^2+1)$$

is a square triangular number, showing that its square root $f(x) = 2x\sqrt{8x^2+1}$ is a balancing number. Again

$$8[g(x)]^2 + 1 = \left(8x + 3\sqrt{8x^2+1}\right)^2$$

shows that $g(x)$ is also a balancing number. Further, since $g\big(g(x)\big) = h(x)$ and $g\big(f(x)\big) = p(x)$, it follows that $h(x)$ and $p(x)$ are also balancing numbers. ∎

2.4.2 Remark. We observe that $f(x)$ always generates even balancing numbers whereas $p(x)$ always generates odd balancing numbers. But when x is odd $g(x)$ is even and when x is even $g(x)$ is odd.

Precisely speaking, for a given balancing number x, there are infinitely many functions generating balancing numbers. It is interesting to identify the function that generates the next balancing number. Since for any balancing number x, $f(x)$ is always even and $p(x)$ is always odd, they cannot generate the next balancing number for any given balancing number x. But balancing numbers, by virtue of the recurrence relation (9), are alternatively odd and even, and for an odd balancing number x, $g(x)$ is even, and for an even balancing number x, $g(x)$ is odd. Now a natural question is "Does $g(x)$ generate the next balancing number for any given balancing number x ?" The answer to this question is affirmative.

2.4.3 Theorem. *For any balancing number* x, $g(x) = 3x + \sqrt{8x^2 + 1}$ *is the balancing number next to it and* $\tilde{g}(x) = 3x - \sqrt{8x^2 + 1}$ *is the balancing number just prior to* x.

Proof. Since x is a balancing number, $x = B_n$ for some positive integer n. The recurrence relation (9) tells us that the next balancing number B_{n+1} is given by

$$\begin{aligned} B_{n+1} &= 6B_n - B_{n-1} \\ &= 3B_n + (3B_n - B_{n-1}). \end{aligned} \qquad \cdots (10)$$

Applying Theorem 2.3.2, and the fact that

$$\frac{B_{n+1}+B_{n-1}}{B_n} = 6$$

to the square of the term inside parenthesis in equation (10), we get

$$\begin{aligned} (3B_n - B_{n-1})^2 &= 9B_n^2 + B_{n-1}^2 - 6B_nB_{n-1} \\ &= 9B_n^2 + B_{n-1}^2 - B_{n-1}(B_{n+1}+B_{n-1}) \\ &= 9B_n^2 - (B_n^2 - 1) \\ &= 8B_n^2 + 1. \end{aligned}$$

Inserting this result in (10), we get

$$B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1},$$

which is equivalent to

$$B_{n+1} = g(B_n),$$

proving that the balancing number next to x is $g(x)$.

Since,

$$g\left(3B_n - \sqrt{8B_n^2+1}\right) = 3\left(3B_n - \sqrt{8B_n^2+1}\right) + \sqrt{8\left(3B_n - \sqrt{8B_n^2+1}\right)^2 + 1}$$

$$= 9B_n - 3\sqrt{8B_n^2+1} + 3\sqrt{8B_n^2+1} - 8B_n$$

$$= B_n,$$

it follows that, B_n is the balancing number next to $3B_n - \sqrt{8B_n^2+1}$. Thus the balancing number prior to B_n is

$$\breve{g}(B_n) = 3B_n - \sqrt{8B_n^2+1}. \qquad \blacksquare$$

2.4.4 Remark. It is well known that, if B is a balancing number, then $8B^2+1$ is a perfect square, and the number

$$C = \sqrt{8B^2+1}$$

occurs frequently in the nonlinear recurrence relations for balancing numbers. For example,

$$B_{n+1} = 3B_n + \sqrt{8B_n^2+1},$$

$$B_{n-1} = 3B_n - \sqrt{8B_n^2+1},$$

by virtue of Theorem 2.4.3, it can be verified that

$$B_{n+2} = 17B_n + 6\sqrt{8B_n^2+1}$$

and so on. We call this number C, the Lucas-balancing number corresponding to the balancing number B. In general, we call

$$C_n = \sqrt{8B_n^2+1},$$

the n^{th} Lucas-balancing number. In Theorem 2.4.7, we will prove below that these numbers are associated with balancing numbers, in the way Lucas numbers are associated with Fibonacci numbers, and hence justifying the name.

It is interesting to note that the sequence of Lucas-balancing numbers satisfy the same recurrence relation as that of balancing numbers.

2.4.5 Theorem. *The sequence of Lucas-balancing numbers satisfies recurrence relation identical to that for balancing numbers. More precisely,* $C_1 = 3,\ C_2 = 17$, and $C_{n+1} = 6C_n - C_{n-1}$.

Proof. From the definition of Lucas-balancing numbers

$$\begin{aligned} C_{n+1}^2 &= 8B_{n+1}^2 + 1 \\ &= 8\left(3B_n + \sqrt{8B_n^2 + 1}\right)^2 + 1 \\ &= \left(3\sqrt{8B_n^2 + 1} + 8B_n\right)^2 \\ &= (3C_n + 8B_n)^2. \end{aligned}$$

Hence,

$$C_{n+1} = 3C_n + 8B_n. \qquad \cdots(11)$$

Similarly, using Theorem 2.4.3, it can be easily seen that

$$C_{n-1} = 3C_n - 8B_n. \qquad \cdots(12)$$

Adding (11) and (12), we get

$$C_{n+1} + C_{n-1} = 6C_n \qquad \cdots(13)$$

from which the desired recurrence relation follows. ■

2.4.6 Remark. Using the recurrence relation (13), it is easy to see that the Binet formula for C_n is

$$C_n = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2}. \qquad \cdots(14)$$

Theorem 2.3.4 provides a method of expressing a balancing number as a function of the balancing number just prior to it. The following theorem provides a method of expressing a balancing number in terms of lower ordered balancing and Lucas-balancing numbers.

2.4.7 Theorem [45]. *If m and n are two positive integers, then* $B_{m+n} = B_m C_n + C_m B_n$.

Proof. By virtue of the Binet formulas for B_n and C_n from (8) and (14),

$$C_n + \sqrt{8} B_n = \alpha_1^{2n}, n = 1, 2, \cdots .$$

Hence,

$$C_{m+n} + \sqrt{8} B_{m+n} = \alpha_1^{2(m+n)}$$

and

$$\left(C_m + \sqrt{8} B_m\right)\left(C_n + \sqrt{8} B_n\right) = \alpha_1^{2m} \alpha_1^{2n} = \alpha_1^{2(m+n)}.$$

Equating the left hand sides of the last two equations, we get

$$(C_m C_n + 8 B_m B_n) + \sqrt{8}(B_m C_n + C_m B_n) = C_{m+n} + \sqrt{8} B_{m+n}.$$

Comparing the rational and irrational parts from both sides, we obtain

$$C_{m+n} = C_m C_n + 8 B_m B_n,$$

and

$$B_{m+n} = B_m C_n + C_m B_n. \qquad \blacksquare$$

Observe that when $m = 1$, Theorem 2.4.7 reduces to Theorem 2.4.3.

The following corollary, the proof of which is contained in the proof of the above theorem, expresses a Lucas-balancing number, in terms of lower ordered balancing and Lucas-balancing numbers.

2.4.8 Corollary. *If m and n are two natural numbers, then* $C_{m+n} = C_m C_n + 8 B_m B_n$.

When $m = n$ in Theorem 2.4.7 and Corollary 2.4.8, we have the following result.

2.4.9 Corollary. *If n is a positive integer, then* $B_{2n} = 2 B_n C_n$ *and* $C_{2n} = C_n^2 + 8 B_n^2$.

If $m + n$ is replaced by $m - n$ $(m > n)$ in Theorem 2.4.7 and Corollary 2.4.8, we get the following result.

2.4.10 Theorem. *If* m *and* n *are two positive integers and* $m > n$, *then* $B_{m-n} = B_m C_n - C_m B_n$ *and* $C_{m-n} = C_m C_n - 8B_m B_n$.

Proof. The proof of

$$B_{m+n} = B_m C_n + C_m B_n$$

in Theorem 2.4.7 is based on the fact

$$C_n + \sqrt{8} B_n = {\alpha_1}^{2n}.$$

Using

$$C_n - \sqrt{8} B_n = {\alpha_2}^{2n},$$

the proof can be completed in a manner similar to that of Theorem 2.4.7. ■

A weaker form of Theorem 2.4.7 gives a function of two variables generating balancing numbers.

2.4.11 Corollary [7]. *If* x *and* y *are balancing numbers then* $f(x,y) = x\sqrt{8y^2+1} + y\sqrt{8x^2+1}$ *is also a balancing number.*

Corollary 2.4.11 can be extended to a function of three variables generating balancing numbers.

2.4.12 Theorem[7]. *If* x, y *and* z *are balancing numbers then* $f(x,y,z) = x\sqrt{8y^2+1}\sqrt{8z^2+1} + y\sqrt{8x^2+1}\sqrt{8z^2+1} + z\sqrt{8x^2+1}\sqrt{8y^2+1} + 8xyz$ *is also a balancing number.*

Proof. Let m, n and r be any three positive integers. Applying Theorem 2.4.7 and Corollary 2.4.8 to B_{m+n+r}, we get

$$\begin{aligned} B_{m+n+r} &= B_m C_{n+r} + C_m B_{n+r} \\ &= B_m (C_n C_r + 8B_n B_r) + C_m (B_n C_r + C_n B_r). \end{aligned} \qquad \cdots (15)$$

Now putting

$$B_m = x, B_n = y, B_r = z$$

and hence,

$$C_m = \sqrt{8x^2+1},\ C_n = \sqrt{8y^2+1},\ C_r = \sqrt{8z^2+1}$$

in (15), we see that the right hand side of (15) is equal to $f(x,y,z)$. ■

Theorem 2.3.2 can be rewritten as

$$(B_n + 1)(B_n - 1) = B_{n+1}B_{n-1}.$$

Observing that $B_1 = 1,$ this result can be further generalized.

2.4.13 Theorem *If n and r are natural numbers and $n > r,$ then $B_{n+r}B_{n-r} = (B_n + B_r)(B_n - B_r).$*

Proof. For any two natural numbers n and r with $n > r,$ by Theorem 2.4.7 we have

$$B_{n+r} = B_nC_r + C_nB_r$$

and by Theorem 2.4.8 we have

$$B_{n-r} = B_nC_r - C_nB_r.$$

Now,

$$\begin{aligned} B_{n+r}B_{n-r} &= (B_nC_r + C_nB_r)(B_nC_r - C_nB_r) \\ &= B_n^2C_r^2 - C_n^2B_r^2 \\ &= B_n^2(8B_r^2 + 1) - (8B_n^2 + 1)B_r^2 \\ &= B_n^2 - B_r^2 \\ &= (B_n + B_r)(B_n - B_r). \end{aligned}$$

■

Some functions in this section look like trigonometric functions and hyperbolic trigonometric functions. For example, the identity

$$B_{m+n} = B_mC_n + C_mB_n$$

resembles

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

or

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y.$$

Indeed, writing $C_m = \cosh x,$ and using the identity

$$\cosh^2 x - \sinh^2 x = 1,$$

it can be easily seen that $\sinh x = \sqrt{8}B_m$. Now taking $C_n = \cosh y$ for some $y,$ and hence, $\sinh y = \sqrt{8}B_n,$ the identities in Theorem 2.4.7, Corollary 2.4.8, Corollary 2.4.9 and Theorem 2.4.10 can be easily verified.

2.5 GENERATING FUNCTION FOR BALANCING NUMBERS

In the Section 2.2, we have got the Binet formula for balancing numbers which is non-recursive. In Section 2.3 and 2.4, we have obtained the recurrence relations for balancing numbers, both linear and nonlinear, and of first and second order. In this section, we obtain another non-recursive form for balancing numbers using the generating function.

2.5.1 Theorem. *The generating function for the sequence of balancing numbers* $\{B_n\}_{n=1}^{\infty}$ *is* $g(s) = \frac{s}{1-6s+s^2}$, *and consequently,* B_{n+1} *is given by*

$$B_{n+1} = 6^n - \binom{n-1}{1}6^{n-2} + \binom{n-2}{2}6^{n-4} - \cdots + (-1)^{\left[\frac{n}{2}\right]}\binom{n-\left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right]}6^{n-2\left[\frac{n}{2}\right]}$$

$$= \sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n-k}{k}6^{n-2k}, \qquad \cdots (16)$$

where $[\cdot]$ *denotes the greatest integer function.*

Proof. Multiplying both sides of the recurrence relation (9) by s^n and taking summation over $n = 1$ to $n = \infty$, we obtain

$$\frac{1}{s}\sum_{n=1}^{\infty} B_n\, s^{n+1} - 6\sum_{n=1}^{\infty} B_n\, s^n + s\sum_{n=1}^{\infty} B_{n-1}\, s^{n-1} = 0.$$

Since $B_0 = 0$, the last equation in terms of $g(s)$ becomes

$$\frac{1}{s}(g(s) - s) - 6g(s) + sg(s) = 0.$$

This gives,

$$g(s) = \frac{s}{1-6s+s^2}.$$

The power series expansion for $g(s)$ is

$$g(s) = s[1 + (6s - s^2) + (6s - s^2)^2 + \cdots];$$

when n is even, the coefficient of s^{n+1} in $g(s)$ is

$$6^n - \binom{n-1}{1}6^{n-2} + \binom{n-2}{2}6^{n-4} - \cdots + (-1)^{n/2}, \qquad \cdots (17)$$

and when n is odd, the coefficient of s^{n+1} in $g(s)$ is

$$6^n - \binom{n-1}{1}6^{n-2} + \binom{n-2}{2}6^{n-4} - \cdots + (-1)^{(n-1)/2}\binom{\frac{n+1}{2}}{\frac{n-1}{2}}6. \qquad \cdots (18)$$

It is clear that (17) represents the right hand side of (16) when n is even, and (18) represents the right hand side of (16) when n is odd. ■

2.6 AN APPLICATION OF BALANCING NUMBERS TO A DIOPHANTINE EQUATION

As discussed in Chapter 1, the general solution of the Diophantine equation $x^2 + y^2 = z^2$, where $x, y, z \in \mathbb{Z}^+$, also known as the Pythagorean equation, is given by

$$x = k(u^2 - v^2),\ y = 2kuv,\ z = k(u^2 + v^2),$$

where $k, u, v \in \mathbb{Z}^+$ [25, 43] such that $u > v$. In this section we consider the particular case

$$x^2 + (x+1)^2 = y^2, \qquad \cdots(19)$$

and obtain the solutions in terms of balancing numbers.

The equation (19) can be rewritten as

$$2y^2 - 1 = (2x+1)^2,$$

and this implies

$$y^2(2y^2 - 1) = \frac{(2y^2-1)2y^2}{2}$$
$$= y^2(2x+1)^2.$$

Since both y^2 and $2y^2 - 1$ are odd, $y^2(2y^2 - 1)$ is an odd square triangular number. Hence $y\sqrt{2y^2 - 1}$ is an odd balancing number, say

$$B = y\sqrt{2y^2 - 1}.$$

Then

$$y^2 = \frac{1+\sqrt{8B^2+1}}{4}, \qquad \cdots(20)$$

so that

$$y = \frac{\sqrt{1+\sqrt{8B^2+1}}}{2}.$$

Substituting (20) into (19) we get,

$$2x^2 + 2x + 1 = \frac{1+\sqrt{8B^2+1}}{4}.$$

Solving the last equation for x we find

$$x = \frac{\sqrt{\frac{1}{2}\left(\sqrt{8B^2+1}-1\right)}-1}{2}.$$

To slightly simplify the expressions for x and y we may write

$$C = \sqrt{8B^2 + 1}$$

for the Lucas-balancing number corresponding to the balancing number B. Thus, the totality of solutions of (19) can be expressed as

$$x = \frac{\sqrt{\frac{1}{2}(C-1)}-1}{2}$$

and

$$y = \frac{\sqrt{1+C}}{2}.$$

The above discussion proves the following theorem.

2.6.1 Theorem. *The solutions of the Diophantine equation* $x^2 + (x+1)^2 = y^2$ *are given by* $x = \frac{\sqrt{\frac{1}{2}(C-1)}-1}{2}$ *and* $y = \frac{\sqrt{1+C}}{2}$, *where* C *is the Lucas-balancing number corresponding to any odd balancing number* B, *that is* $C = \sqrt{8B^2 + 1}$.

2.6.2 Examples. The first odd balancing number is 1, and the corresponding Lucas-balancing number is 3. Using Theorem 2.6.1, the solutions of

$$x^2 + (x+1)^2 = y^2$$

are given by

$$x = \frac{\sqrt{\frac{1}{2}(3-1)}-1}{2} = 0$$

and

$$y = \frac{\sqrt{1+3}}{2} = 1$$

leading to

$$0^2 + 1^2 = 1^2$$

which is trivial and of no interest. The second odd balancing number 35, corresponds to the Lucas-balancing number 99, and we have

$$x = \frac{\sqrt{\frac{1}{2}(99-1)}-1}{2} = 3$$

and

$$y = \frac{\sqrt{1+99}}{2} = 5$$

leading to

$$3^2 + 4^2 = 5^2.$$

Similarly, $B = 1189$ and hence, $C = 3363$ correspond to the solution

$$x = \frac{\sqrt{\frac{1}{2}(3363-1)}-1}{2} = 20$$

and

$$y = \frac{\sqrt{1+3363}}{2} = 29$$

leading to

$$20^2 + 21^2 = 29^2.$$

We observe that the values of y in the solution of

$$x^2 + (x+1)^2 = y^2$$

in the above examples are all odd Pell numbers. In Chapter 4, we relate the solutions of the above equation with Pell numbers.

CHAPTER 3

Cobalancing Numbers

3.1 INTRODUCTION

As discussed in Chapter 2, a natural number n is called a balancing number if it satisfies the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \qquad \cdots (1)$$

for some natural number r, where r is called the balancer corresponding to the balancing number n.

By slightly modifying equation (1), we call a natural number n, a *cobalancing number* if

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) \qquad \cdots (2)$$

for some natural number r. We call r, the *cobalancer* corresponding to the cobalancing number n.

The first three cobalancing numbers are 2, 14 and 84 with cobalancers 1, 6 and 35 respectively.

It is clear from (2) that, n is a cobalancing number with cobalancer r if and only if

$$n(n+1) = \frac{(n+r)(n+r+1)}{2},$$

which, when solved for r gives

$$r = \frac{-(2n+1)+\sqrt{8n^2+8n+1}}{2}. \qquad \cdots (3)$$

It follows from (3) that n is a cobalancing number if and only if $8n^2 + 8n + 1$ is a perfect square, that is, $n(n+1)$ is a triangular number. Since $8 \cdot 0^2 + 8 \cdot 0 + 1 = 1$ is a perfect square, we accept 0 as a cobalancing number, just like Behera and Panda [7] accepted 1 as a balancing number, though, by definition, a cobalancing number should be greater than 1. Thus, denoting the n^{th} cobalancing number by b_n, we have $b_1 = 0$, $b_2 = 2$, $b_3 = 14$ and so on.

From the above discussion, it is clear that, if n is a cobalancing number, then both $n(n+1)$ and $n(n+1)/2$ are triangular numbers. Thus, our search for cobalancing numbers is confined to the triangular numbers that are also pronic. Since for $n \geq 1$, $n < \sqrt{n(n+1)} < n+1$, it follows that, for any pronic triangular number T, $\left[\sqrt{T}\,\right]$ is a cobalancing number, where $[\cdot]$ denote the greatest integer function. For example, $T = 6$ is a pronic triangular number and therefore, $\left[\sqrt{6}\,\right] = 2$ is a cobalancing number.

In Chapter 2, we have seen that, a natural number n is a balancing number if and only if n^2 is a triangular number, and we have just proved that, n is a cobalancing number if and only if $n(n+1)$ is a triangular number. Since $n(n+1)$ is a pronic number, the cobalancing numbers are associated with the pronic triangular numbers, just like balancing numbers are associated with the square triangular numbers. Figure 3.1 demonstrates the figurate number representation of the pronic triangular numbers 6:

Figure 3.1: The pronic triangular number 6 as triangle and rectangle; side of the triangle $b + r = 3$, smaller side of the rectangle $b = 2$.

Indeed, pronic numbers are also related to perfect squares in the sense that they are almost equal to the arithmetic mean of squares of two consecutive natural numbers, that is, $n(n+1) \approx [n^2 + (n+1)^2]/2$.

In [51, 52, 53], Subramaniam has explored some interesting properties of square triangular numbers. In a latter paper [54], he introduced the concept of almost square triangular numbers (triangular numbers differing from a square by unity), and established links with the square triangular numbers without explicitly linking to balancing numbers, though, linking to square triangular numbers automatically creates links to balancing numbers. In Section 3.6, we establish a direct link between balancing and cobalancing numbers, and hence between square triangular numbers and pronic triangular numbers.

3.2 SOME FUNCTIONS OF COBALANCING NUMBERS

In this section, we introduce some functions of cobalancing numbers that also generate cobalancing numbers. For any two cobalancing numbers x and y, we consider the following functions:

$$f(x) = 3x + \sqrt{8x^2 + 8x + 1} + 1$$

$$g(x) = 17x + 6\sqrt{8x^2 + 8x + 1} + 8$$

$$h(x) = 8x^2 + 8x + 1 + (2x+1)\sqrt{8x^2 + 8x + 1} + 1$$

$$\begin{aligned} t(x, y) = \tfrac{1}{2}\Big[&2(2x+1)(2y+1) + (2x+1)\sqrt{8y^2 + 8y + 1} \\ &+(2y+1)\sqrt{8x^2 + 8x + 1} \\ &+\sqrt{8x^2 + 8x + 1}\sqrt{8y^2 + 8y + 1} - 1\Big] \end{aligned}$$

We first prove that the above functions always generate cobalancing numbers.

3.2.1 Theorem. *For any two cobalancing numbers* x *and* y, $f(x)$, $g(x)$, $h(x)$ *and* $t(x,y)$ *are all cobalancing numbers.*

Proof. Suppose that $u = f(x)$. Then $x < u$ and

$$x = 3u - \sqrt{8u^2 + 8u + 1} + 1.$$

Since x and u are non-negative integers, $8u^2 + 8u + 1$ must be a perfect square, and hence u is a cobalancing number. Since $f\big(f(x)\big) = g(x)$, it follows that $g(x)$ is also a cobalancing number. It can also be directly verified that $8h^2(x) + 8h(x) + 1$ and $8t^2(x,y) + 8t(x,y) + 1$ are perfect squares so that $h(x)$ and $t(x,y)$ are cobalancing numbers. But these verifications would involve lengthy algebra. To avoid algebraic complications, we provide relatively easy proofs of these results in Section 3.6. ■

Next, we show that for any cobalancing number x, $f(x)$ is not merely a cobalancing number, but it is the cobalancing number next to x.

3.2.2 Theorem. *If* x *is any cobalancing number, then the cobalancing number next to* x *is* $f(x) = 3x + \sqrt{8x^2 + 8x + 1} + 1$, *and consequently, the previous one is* $\tilde{f}(x) = 3x - \sqrt{8x^2 + 8x + 1} + 1$.

Proof. The function $f: [0, \infty) \to [2, \infty)$, defined by

$$f(x) = 3x + \sqrt{8x^2 + 8x + 1} + 1$$

is strictly increasing since

$$f'(x) = 3 + \frac{4(x+1)}{\sqrt{8x^2+8x+1}} > 0.$$

Also f is one-to-one and $x < f(x)$ for all $x \geq 0$. This shows that f^{-1} exists and is also strictly increasing with $f^{-1}(x) < x$. It is easy to prove that

$$f^{-1}(x) = 3x - \sqrt{8x^2 + 8x + 1} + 1$$

and since

$$8[f^{-1}(x)]^2 + 8f^{-1}(x) + 1 = \left(3\sqrt{8x^2 + 8x + 1} - 8x - 4\right)^2,$$

it follows that $f^{-1}(x)$ is also a cobalancing number. We complete the remaining part of the proof by induction.

The first three cobalancing numbers are $b_1 = 0,\ b_2 = 2,\ b_3 = 14$ and satisfy the relationship $f(b_1) = b_2$, and $f(b_2) = b_3$. Let assume that there is no cobalancing number between b_{n-1} and b_n for $n = 1,2,\cdots,k$. To complete the induction, we need to prove that there is no cobalancing number between b_k and b_{k+1}. Let us assume to the contrary that there exists a cobalancing number y between b_k and b_{k+1}. Then

$$b_k < y < b_{k+1}$$

implies

$$f^{-1}(b_k) < f^{-1}(y) < f^{-1}(b_{k+1})$$

which, in turn, implies

$$b_{k-1} < f^{-1}(y) < b_k.$$

Since y is a cobalancing number, by Theorem 3.2.2, $f^{-1}(y)$ is also a cobalancing number, and the existence of a cobalancing number between b_{k-1} and b_k is a contradiction to our assumption that there is no cobalancing number between b_{k-1} and b_k. ∎

3.3 RECURRANCE RELATIONS FOR COBALANCING NUMBERS

For $n = 1, 2, \cdots$, as usual, let b_n be the n^{th} cobalancing number. Theorem 3.2.2 suggests

$$b_{n+1} = 3b_n + \sqrt{8{b_n}^2 + 8b_n + 1} + 1,$$

and

$$b_{n-1} = 3b_n - \sqrt{8{b_n}^2 + 8b_n + 1} + 1.$$

Adding the last two equations, we arrive at the conclusion that, the cobalancing numbers obey the second-order linear recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2, n \geq 2. \qquad \cdots(4)$$

An immediate consequence of (4) is the following:

3.3.1 Theorem. *Every cobalancing number is even.*

Proof. The proof is based on mathematical induction. The first two cobalancing numbers $b_1 = 0$ and $b_2 = 2$ are even. Assuming that b_n is even for $n \le k$ and using (4) one can easily see that b_{k+1} is also even. ■

Using the recurrence relation (4), we can derive some other interesting relations among the cobalancing numbers.

Theorem 2.3.2 suggests that the balancing numbers satisfy

$$B_n^2 = 1 + B_{n-1}B_{n+1}, n \ge 2.$$

We can expect a similar result for cobalancing numbers too. The following theorem demonstrates this result.

3.3.2 Theorem. *The cobalancing numbers satisfy the second order nonlinear recurrence relation* $(b_n - 1)^2 = 1 + b_{n-1}b_{n+1}, n \ge 2.$

Proof. Using (4) we get

$$\frac{b_{n+1}+b_{n-1}-2}{b_n} = 6.$$

Replacing n by $n-1$, we obtain

$$\frac{b_n+b_{n-2}-2}{b_{n-1}} = 6,$$

which gives

$$\frac{b_{n+1}+b_{n-1}-2}{b_n} = \frac{b_n+b_{n-2}-2}{b_{n-1}},$$

and on further rearrangement, it gives

$$(b_n - 1)^2 - b_{n-1}b_{n+1} = (b_{n-1} - 1)^2 - b_{n-2}b_n.$$

Now iterating recursively we obtain

$$\begin{aligned}(b_n - 1)^2 - b_{n-1}b_{n+1} &= (b_2 - 1)^2 - b_1b_3 \\ &= (2-1)^2 - 0 \times 14 = 1,\end{aligned}$$

from which the assertion follows. ■

The following theorem provides a link between a cobalancing number of certain order and balancing and cobalancing numbers of lower order.

3.3.3 Theorem. *If* n *and* k *are natural numbers such that* $n > k \geq 2$, *then* $b_n = b_k + B_k b_{n-k+1} - B_{k-1} b_{n-k}$.

The proof of Theorem 3.3.3 needs an important link between balancing and cobalancing numbers, to be established in the next section after Theorem 3.4.1. Until then, we postpone the proof of this theorem.

The following corollary establishes relationships among the cobalancing numbers of even and odd order with balancing and cobalancing numbers of lower order.

3.3.4 Corollary. *If* n *is natural number then* $b_{2n} = B_n b_{n+1} - b_n(B_{n-1} - 1)$ *and* $b_{2n-1} = (B_n + 1)b_n - B_{n-1} b_{n-1}$.

Proof. The proof of the first part follows from Theorem 3.3.3 replacing n by $2n$ and k by n. Similarly the proof of the second part follows from Theorem 3.3.3 replacing n by $2n - 1$ and k by n. ∎

If B is a balancing number, then it is well known that $8B^2 + 1$ is a perfect square and

$$C = \sqrt{8B^2 + 1}$$

is called a Lucas-balancing number. In the same way, if b is a cobalancing number then $8b^2 + 8b + 1$ is a perfect square, and we call

$$c = \sqrt{8b^2 + 8b + 1}$$

a Lucas-cobalancing number. In general, we write

$$c_n = \sqrt{8b_n^2 + 8b_n + 1}$$

for the n^{th} Lucas-cobalancing number. Thus, $c_1 = 1$, $c_2 = 7$ and so on. In Chapter 2, we have observed that the Lucas-balancing numbers obey the recurrence relation identical to that for balancing numbers. Thus, we can expect the Lucas-cobalancing numbers to obey the same recurrence relation as that for cobalancing numbers. But unfortunately, this is not true; rather the Lucas-cobalancing numbers obey the recurrence relation satisfied by balancing numbers.

3.3.5 Theorem. *The sequence of Lucas-cobalancing numbers satisfies recurrence relation identical with that for balancing numbers. More precisely,* $c_1 = 1$, $c_2 = 7,\ c_{n+1} = 6c_n - c_{n-1}$ *for* $n = 2, 3, \cdots$.

Proof. We have

$$\begin{aligned} c_{n+1}^2 &= 8b_{n+1}^2 + 8b_{n+1} + 1 \\ &= 8\left(3b_n + \sqrt{8{b_n}^2 + 8b_n + 1} + 1\right)^2 + 8b_{n+1} + 1 \\ &= \left(3\sqrt{8{b_n}^2 + 8b_n + 1} + 8b_n + 4\right)^2 \\ &= (3c_n + 8b_n + 4)^2. \end{aligned}$$

Hence,

$$c_{n+1} = 3\,c_n + 8b_n + 4. \qquad \cdots(5)$$

Using

$$c_{n-1}^2 = 8b_{n-1}^2 + 8b_{n-1} + 1$$

in a in a similar way, it can be proved that

$$c_{n-1} = 3\,c_n - 8b_n - 4. \qquad \cdots(6)$$

Adding (5) and (6), we obtain

$$c_{n+1} + c_{n-1} = 6c_n,$$

from which the assertion of the theorem follows. ■

3.3.6 Remark. Using the above recurrence relation for c_n, it can be easily seen that the Binet formula for c_n is

$$c_n = \frac{{\alpha_1}^{2n-1} + {\alpha_2}^{2n-1}}{2}, n = 1, 2, \cdots. \qquad \cdots(7)$$

3.4 GENERATING FUNCTION FOR COBALANCING NUMBERS

In Section 3.3, we have developed the recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2, n \geq 2$$

for cobalancing numbers. Using this recurrence relation, we first obtain the generating function for cobalancing numbers, and then establish a very interesting link between balancing and cobalancing numbers.

3.4.1 Theorem. *The generating function for the sequence of cobalancing numbers* $\{b_n\}_{n=1}^{\infty}$ is

$$f(s) = \frac{2s^2}{(1-s)(1-6s+s^2)}, \qquad \cdots (8)$$

and consequently, for $n \geq 2, b_n = 2(B_1 + B_2 + \cdots + B_{n-1})$.

Proof. Equation (4) can be rewritten as

$$b_{n+2} - 6b_{n+1} + b_n = 2.$$

Multiplying both sides by s^{n+2} and summing over $n = 1$ to $n = \infty$, we obtain

$$\sum_{n=1}^{\infty} b_{n+2}\, s^{n+2} - 6s \sum_{n=1}^{\infty} b_{n+1}\, s^{n+1} + s^2 \sum_{n=1}^{\infty} b_n\, s^n = 2s^2 \sum_{n=1}^{\infty} s^n,$$

which, in terms of $f(s)$, can be expressed as

$$(f(s) - 2s^2) - 6sf(s) + s^2 f(s) = \frac{2s^3}{1-s}.$$

Thus,

$$\begin{aligned} f(s) &= \frac{2s^2}{(1-s)(1-6s+s^2)} \\ &= \frac{2s}{1-s} \cdot \frac{s}{1-6s+s^2} = \frac{2s}{1-s} \cdot g(s) \\ &= 2(s + s^2 + \cdots) g(s), \end{aligned}$$

where $g(s)$ is the generating function for balancing numbers discussed in Chapter 2. For $n \geq 2$, the coefficient of s^n in $f(s)$ can be obtained by collecting the coefficient of s^r from $g(s)$ and the coefficient of s^{n-r} from $2(s + s^2 + \cdots)$, $r = 1, 2, \cdots, n-1$. While the coefficient of s^r in $g(s)$ is B_r, the coefficient of s^{n-r} in $2(s + s^2 + \cdots)$ is 2. Hence,

$$b_n = 2(B_1 + B_2 + \cdots + B_{n-1}).$$

This completes the proof. ■

The following corollary and Theorem 3.3.1 are direct consequences of Theorem 3.4.1.

3.4.2 Corollary. *The* n^{th} *balancing number is half the difference of the* $(n+1)^{st}$ *and the* n^{th} *cobalancing numbers. More precisely,* $B_n = \frac{b_{n+1} - b_n}{2}$.

Proof. Since

$$b_n = 2(B_1 + B_2 + \cdots + B_{n-1})$$

by Theorem 3.4.1, we have

$$b_{n+1} - b_n = 2B_n$$

from which the result follows. ■

We are now in a position to prove Theorem 3.3.3.

3.4.3 Proof of Theorem 3.3.3. The proof is based on induction on k. In view of (4), it is easy to see that the assertion is true for $n > k = 2$. Assume that the assertion is true for $n > r \geq k \geq 2$, that is,

$$b_n = b_k + B_k b_{n-k+1} - B_{k-1} b_{n-k} \qquad \cdots (9)$$

for $k \leq r$. In Chapter 2, we have seen that, the balancing numbers obey the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}.$$

Applying this relation, (4), (9) for $k = r$ and Corollary 3.4.2 to the right hand side of (9) with $k = r + 1$, we obtain

$$\begin{aligned}
&b_{r+1} + B_{r+1} b_{n-r} - B_r b_{n-r-1} \\
&= b_{r+1} + (6B_r - B_{r-1}) b_{n-r} - B_r(6b_{n-r} - b_{n-r+1} + 2) \\
&= b_{r+1} - 2B_r + B_r b_{n-r+1} - B_{r-1} b_{n-r} \\
&= b_r + B_r b_{n-r+1} - B_{r-1} b_{n-r} \\
&= b_n.
\end{aligned}$$

Thus the assertion is also true for $k = r + 1$. This completes the proof of Theorem 3.3.3. ■

Theorem 3.3.3 can be stated in a more simplified form as follows:

3.4.4 Corollary. *If* m *and* n *are two natural numbers, then* $b_{m+n} = b_m + B_m b_{n+1} - B_{m-1} b_n$.

3.5 BINET FORMULA FOR COBALANCING NUMBERS

As usual, we write $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$. In Section 3.4, we have seen that the cobalancing numbers satisfy the recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2;\ n \geq 2,$$

which is a second order linear non-homogeneous difference equation with constant coefficients. Substituting

$$d_n = b_n + \frac{1}{2},$$

we see that d_n obey the recurrence relation

$$d_{n+1} = 6d_n - d_{n-1}$$

which is homogeneous. The general solution of this equation is

$$d_n = A\lambda_1^n + B\lambda_2^n \qquad \cdots(10)$$

where

$$\lambda_1 = 3 + \sqrt{8} = \alpha_1^2,$$

and

$$\lambda_2 = 3 - \sqrt{8} = \alpha_2^2$$

are the two roots of the auxiliary equation

$$\lambda^2 - 6\lambda + 1 = 0.$$

Substituting $d_1 = \frac{1}{2}$ and $d_2 = \frac{5}{2}$ into (10), we obtain

$$A = \frac{1}{\alpha_1(\lambda_1 - \lambda_2)},$$

and

$$B = \frac{1}{\alpha_2(\lambda_1 - \lambda_2)}.$$

Thus,

$$d_n = A\lambda_1^n + B\lambda_2^n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}};\ n = 1,2,\cdots$$

from which we get

$$b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2};\quad n = 1,2,\cdots .$$

Thus, we have the following theorem:

3.5.1 Theorem. *For* $n = 1,2,\cdots$, *the Binet formula for cobalancing numbers is given by* $b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$.

3.6 RELATIONS AMONG BALANCING AND SOME ASSOCIATED NUMBERS

Let B be a balancing number with balancer R and b, a cobalancing number with cobalancer r. By definition, the pairs (B, R) and (b, r) satisfy respectively

$$1 + 2 + \cdots + (B-1) = (B+1) + (B+2) + \cdots + (B+R) \qquad \cdots (11)$$

and

$$1 + 2 + \cdots + b = (b+1) + (b+2) + \cdots + (b+r). \qquad \cdots (12)$$

Solving (11) for B and (12) for b we find

$$B = \frac{(2R+1)+\sqrt{8R^2+8R+1}}{2}, \qquad \cdots (13)$$

and

$$b = \frac{(2r-1)+\sqrt{8r^2+1}}{2}. \qquad \cdots (14)$$

We infer from (13) that, if R is a balancer then $8R^2 + 8R + 1$ is a perfect square, and from (14) we conclude that if r is a cobalancer then $8r^2 + 1$ is a perfect square.

The above discussion proves the following theorem:

3.6.1 Theorem. *Every balancer is a cobalancing number and every cobalancer is a balancing number.*

For $n = 1, 2, \cdots$, as usual, let B_n be the n^{th} balancing number and b_n the n^{th} cobalancing number. We also denote by R_n, the balancer corresponding to B_n and r_n the cobalancer corresponding to b_n. What we are going to prove now is a result which is much stronger than Theorem 3.6.1.

3.6.2 Theorem. *For* $n = 1, 2, \cdots$, *the* n^{th} *balancer is equal to the* n^{th} *cobalancing number and the* $(n+1)^{st}$ cobalancer is equal to the n^{th} balancing number, that is, $R_n = b_n$ *and* $r_{n+1} = B_n$.

Proof. If B is a balancing number with balancer R, then by virtue of the relationship between balancing numbers and balancers,

$$R = \frac{-(2B+1)+\sqrt{8B^2+1}}{2}.$$

Thus,

$$R_{n+1} = \frac{-(2B_{n+1}+1)+\sqrt{8B_{n+1}^2+1}}{2}, \qquad \cdots (15)$$

and

$$R_{n-1} = \frac{-(2B_{n-1}+1)+\sqrt{8B_{n-1}^2+1}}{2}. \qquad \cdots (16)$$

Also, from Theorem 2.4.3,

$$B_{n+1} = 3B_n + \sqrt{8B_n^2+1} \qquad \cdots (17)$$

and

$$B_{n-1} = 3B_n - \sqrt{8B_n^2+1}. \qquad \cdots (18)$$

Substituting (17) and (18) into (15) and (16) respectively, we obtain

$$R_{n+1} = \frac{2B_n+\sqrt{8B_n^2+1}-1}{2}$$

and

$$R_{n-1} = \frac{-14B_n+5\sqrt{8B_n^2+1}-1}{2}.$$

Adding the last two equations, we get

$$\begin{aligned} R_{n+1} + R_{n-1} &= \frac{-12B_n+6\sqrt{8B_n^2+1}-2}{2} \\ &= 6 \cdot \frac{-(2B_n+1)+\sqrt{8B_n^2+1}}{2} + 2 \\ &= 6R_n + 2. \end{aligned}$$

This gives

$$R_{n+1} = 6R_n - R_{n-1} + 2,$$

showing that the sequence $\{R_n\}_{n=1}^{\infty}$ satisfies the same recurrence relation as that for $\{b_n\}_{n=1}^{\infty}$. Further, since

$$R_1 = b_1 = 0$$

and

$$R_2 = b_2 = 2,$$

it follows that $R_n = b_n$ for $n = 1,2,\cdots$. This proves the first part of the theorem.

We prove the second part of the theorem in a similar way. Using (3) we obtain

$$r_{n+1} = \frac{-(2b_{n+1}+1)+\sqrt{8b_{n+1}^2+8b_{n+1}+1}}{2} \qquad \cdots (19)$$

and

$$r_{n-1} = \frac{-(2b_{n-1}+1)+\sqrt{8b_{n-1}^2+8b_{n-1}+1}}{2}. \qquad \cdots (20)$$

Substituting

$$b_{n+1} = 3b_n + \sqrt{8{b_n}^2 + 8b_n + 1} + 1$$

into (19) and

$$b_{n-1} = 3b_n - \sqrt{8{b_n}^2 + 8b_n + 1} + 1$$

into (20), we obtain

$$r_{n+1} = \frac{2b_n+\sqrt{8b_{n+1}^2+8b_{n+1}+1}+1}{2},$$

and

$$r_{n-1} = \frac{-14b_n+5\sqrt{8b_{n-1}^2+8b_{n-1}+1}-7}{2}.$$

Adding the last two equations, we get

$$\begin{aligned} r_{n+1} + r_{n-1} &= \frac{-12b_n+6\sqrt{8b_n^2+8b_n+1}-6}{2} \\ &= 6 \cdot \frac{-(2b_n+1)+\sqrt{8b_n^2+8b_n+1}}{2} = 6r_n. \end{aligned}$$

Thus, the sequence $\{r_n\}_{n=1}^{\infty}$ satisfies the same recurrence relation as that for $\{B_n\}_{n=1}^{\infty}$. Further, since $B_1 = r_2 = 1$ and $B_2 = r_3 = 6$, it follows that $B_n = r_{n+1}$ for $n = 1,2,\cdots$. This completes the proof of the theorem. ■

The following are some consequences of the above theorem.

3.6.3 Corollary. *Every balancer is even.*

Proof. Directly follows from Theorem 3.3.1 and Theorem 3.6.2. ■

3.6.4 Corollary. *The $(n+1)^{st}$ balancer is equal to the sum of n^{th} balancer and twice of the n^{th} balancing number, that is, $R_{n+1} = R_n + 2B_n$.*

Proof. Directly follows from Corollary 3.4.2 and Theorem 3.6.2. ∎

Now, we are in a position to prove that $h(x)$ and $t(x,y)$ are cobalancing numbers as stated in Theorem 3.2.1.

We first show that , if x is a cobalancing number then

$$h(x) = 8x^2 + 8x + 1 + (2x+1)\sqrt{8x^2 + 8x + 1} + 1$$

is also a cobalancing number.

By virtue of Theorem 2.4.1, it follows that, if y is a balancing number, then

$$u = 2y\sqrt{8y^2 + 1}$$

is also a balancing number, and the balancer corresponding to u is

$$\begin{aligned} R &= \frac{-(2u+1)+\sqrt{8u^2+1}}{2} \\ &= 8y^2 - 2y\sqrt{8y^2+1}\,. \end{aligned} \qquad \cdots (21)$$

If x is the balancer corresponding to the balancing number y, then from (10) we find

$$y = \frac{-(2x+1)+\sqrt{8x^2+8x+1}}{2}$$

so that

$$\begin{aligned} 8y^2 + 1 &= 24x^2 + 24x + 4(2x+1)\sqrt{8x^2+8x+1} + 5 \\ &= \left\{2(2x+1) + \sqrt{8x^2+8x+1}\right\}^2. \end{aligned} \qquad \cdots (22)$$

Substitution of (19) into (18) gives

$$\begin{aligned} R &= 24x^2 + 24x + 4(2x+1)\sqrt{8x^2+8x+1} + 4 \\ &\quad -2\left\{\frac{2(2x+1)+\sqrt{8x^2+8x+1}}{2}\right\}\left\{2(2x+1) + \sqrt{8x^2+8x+1}\right\} \\ &= 8x^2 + 8x + 1 + (2x+1)\sqrt{8x^2+8x+1} \\ &= h(x). \end{aligned}$$

Thus, for any balancer x, $h(x)$ is always a balancer. Since by Theorem 3.6.1, every balancer is a cobalancing number, $h(x)$ is a cobalancing number.

We next prove that, if x and y are cobalancing numbers then

$$\begin{aligned} t(x,y) = \tfrac{1}{2}\{2(2x+1)(2y+1) + (2x+1)\sqrt{8y^2+8y+1} \\ +(2y+1)\sqrt{8x^2+8x+1} \\ +\sqrt{8x^2+8x+1}\sqrt{8y^2+8y+1} - 1\} \end{aligned}$$

is also a cobalancing number.

If u and v are balancing numbers, then by Corollary 2.4.11,

$$w = u\sqrt{8v^2+1} + v\sqrt{8u^2+1}$$

is also a balancing number. Let s, x and y be the balancers corresponding to the balancing numbers w, u and v respectively. Then

$$\begin{aligned} s &= \frac{-(2w+1)+\sqrt{8w^2+1}}{2} \\ &= \tfrac{1}{2}\Big[8uv + \sqrt{(8u^2+1)(8v^2+1)} \\ &\quad -2u\sqrt{8v^2+1} - 2v\sqrt{8u^2+1} - 1\Big]. \qquad \cdots (23) \end{aligned}$$

Now substituting

$$u = \frac{(2x+1)+\sqrt{8x^2+8x+1}}{2}$$

and

$$v = \frac{(2y+1)+\sqrt{8y^2+8y+1}}{2}$$

into (20), we find

$$\begin{aligned} s &= \tfrac{1}{2}\Big[2(2x+1)(2y+1) + (2x+1)\sqrt{8y^2+8y+1} \\ &\quad +(2y+1)\sqrt{8x^2+8x+1} \\ &\quad +\sqrt{8x^2+8x+1}\sqrt{8y^2+8y+1} - 1\Big] \\ &= t(x,y). \end{aligned}$$

Again since every balancer is a cobalancing number by Theorem 3.6.1, $t(x,y)$ is a cobalancing number ■

It is interesting to note that $t(x,x) = h(x)$.

3.7 APPLICATIONS OF COBALANCING NUMBERS TO DIOPHANTINE EQUATIONS

In Chapter 2, we have studied the Diophantine equation $x^2+(x+1)^2=y^2$; $x,\ y\in\mathbb{Z}^+$, which is a particular case of the Pythagorean equation $x^2+y^2=z^2$; $x,y,z\in\mathbb{Z}^+$ and linked the solutions with balancing numbers. Here, we are going to provide an easy relationship between the solutions of this equation and cobalancing numbers. We also prove the nonexistence of solution of a Diophantine equation using the evenness of cobalancing numbers.

Let b be any cobalancing number, r its cobalancer and $u=b+r$. Then equation (2) can be re-written as

$$1+2+\cdots+b=(b+1)+(b+2)+\cdots+u,$$

from which, we find b in terms of u as

$$b=-1+\sqrt{2u^2+2u+1}.$$

Thus, $2u^2+2u+1$ is a perfect square and also

$$2u^2+2u+1=u^2+(u+1)^2.$$

This suggests that the Diophantine equation

$$x^2+(x+1)^2=y^2$$

is satisfied by

$$x=u,\ y=\sqrt{2u^2+2u+1}.$$

The above discussion proves the following theorem:

3.7.1 Theorem. *The Pythagorean equation* $x^2+(x+1)^2=y^2$ *is satisfied by* $x=u,\ y=\sqrt{2u^2+2u+1}$, *where* $u=b+r$, b *is a cobalancing number and* r *its cobalancer.*

3.7.2 Examples. If $b = 14$, then $r = 6$ and $u = b + r = 20$. Further

$$\sqrt{2u^2 + 2u + 1} = 841 = 29^2,$$

and we have

$$20^2 + 21^2 = 29^2.$$

Similarly, for $b = 84$, we have

$$119^2 + 120^2 = 169^2.$$

As mentioned at the beginning of this section, the evenness of cobalancing numbers can also be used in proving the non-existence of solution of Diophantine equations. The proof of the following theorem establishes this claim.

3.7.3 Theorem. *The Diophantine equation* $32x^2 - 16x + 1 = y^2$ *has no solution in positive integers.*

Proof. The Diophantine equation

$$32x^2 - 16x + 1 = y^2$$

suggests that y must be odd. Writing $y = 2z + 1$, the above equation reduces to

$$(2x - 1)2x = \frac{z(z+1)}{2},$$

which indicates that the pronic number $(2x - 1)2x$ is also triangular. Hence, by virtue of the discussion in Section 3.1, $2x - 1$ is a cobalancing number. But this a contradiction to Theorem 3.3.1 which states that every cobalancing number is even. Hence, the Diophantine equation

$$32x^2 - 16x + 1 = y^2$$

has no solution in positive integers.

CHAPTER 4

Links of Balancing and Cobalancing Numbers with Pell and Associated Pell Numbers

4.1 INTRODUCTION

The study of number sequences has been a source of attraction to mathematicians since ancient times. From that time, many mathematicians have been focusing their attention on the study of the fascinating triangular numbers. In Chapters 2 and 3, while dealing with equations on triangular numbers, we have observed close association of balancing and cobalancing numbers with square and pronic triangular numbers respectively. In this chapter, we explore some important links of balancing numbers, cobalancing numbers, and other numbers associated balancing and cobalancing numbers with Pell and associated Pell numbers. Further, we study some simple Diophantine equations whose solutions are closely associated with balancing, cobalancing, Pell and associated Pell numbers. Finally, we establish some simple links of balancing, cobalancing and Lucas-balancing numbers with perfect squares.

4.2 SOME IMPORTANT LINKS

In this section, we establish many important connections of balancing, cobalancing and related number sequences with Pell and associated Pell sequences. In particular, we prove the existence of Pell and associated Pell numbers in the factorization of balancing and cobalancing numbers and even in the greatest common divisors of these numbers.

Throughout this section $\alpha_1 = 1+\sqrt{2}$, $\alpha_2 = 1-\sqrt{2}$, and the greatest common divisor of two positive integers m and n is denoted by (m,n). We observe that $\alpha_1\alpha_2 = -1$, and we shall keep on using this result as and when necessary without further mention.

We start with the following important theorem, which gives a direct connection of balancing numbers with Pell and associated Pell numbers.

4.2.1 Theorem. *For $n = 1, 2, \cdots$, the n^{th} balancing number is product of the n^{th} Pell number and the n^{th} associated Pell number, that is $B_n = P_n Q_n$.*

Proof. Using the Binet formulas of P_n, Q_n and B_n, we obtain

$$B_n = \frac{\alpha_1^{2n}-\alpha_2^{2n}}{4\sqrt{2}}$$

$$= \frac{\alpha_1^{n}-\alpha_2^{n}}{2\sqrt{2}} \cdot \frac{\alpha_1^{n}+\alpha_2^{n}}{2} = P_n Q_n. \qquad \blacksquare$$

This is only the basic relationship of balancing numbers with Pell and associated Pell numbers. Truly speaking, the sequences of balancing and cobalancing numbers are contained in a sequence which is the sequence of Pell numbers divided by two, whereas, the sequences of Lucas-balancing and Lucas-cobalancing numbers together constitute the associated Pell sequence.

The following theorem provides a close association of balancing and cobalancing numbers with Pell numbers.

4.2.2 Theorem. *If P is a Pell number, then $[P/2]$ is either a balancing number or a cobalancing number, where $[.]$ denotes the greatest integer function. More specifically, $P_{2n}/2 = B_n$ and $[P_{2n-1}/2] = b_n, n = 1, 2, \cdots$.*

Proof. Using the Binet formulas for P_n, B_n and b_n, we get

$$\frac{P_{2n}}{2} = \frac{\alpha_1^{2n}-\alpha_2^{2n}}{4\sqrt{2}} = B_n,$$

and since P_{2n-1} is odd,

$$\left[\frac{P_{2n-1}}{2}\right] = \frac{P_{2n-1}}{2} - \frac{1}{2}$$

$$= \frac{\alpha_1^{2n-1}-\alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} = b_n. \qquad \blacksquare$$

The following theorem establishes the fact that the union of the sequences of Lucas-balancing and Lucas-cobalancing numbers constitute the sequence of associated Pell numbers.

4.2.3 Theorem. *Every associated Pell number is either a Lucas-balancing or a Lucas-cobalancing number. More specifically,* $Q_{2n} = C_n$ *and* $Q_{2n-1} = c_n, n = 1, 2, \cdots$.

Proof. The proof of the first part immediately follows from the Binet formulas for Q_n and C_n and the proof of the second part follows from the Binet formulas for Q_n and c_n respectively. ■

It is seen in Chapter 2 that, if B is a balancing number with balancer R then $B + R$ represents the side of the triangle in the figurate number representation of the square triangular number B^2 as a triangle. The following theorem demonstrates the association of this number $B + R$ with the partial sums of Pell sequence up to an odd term.

4.2.4 Theorem. *The sum of first* $2n - 1$ *Pell numbers is equal to the sum of* n^{th} *balancing number and its balancer.*

Proof. Using the Binet formulas for P_n, B_n and b_n to the partial sum of first $2n - 1$ Pell numbers, we get

$$\begin{aligned}
P_1 + P_2 + \cdots + P_{2n-1} &= \frac{\alpha_1 - \alpha_2}{2\sqrt{2}} + \frac{\alpha_1^2 - \alpha_2^2}{2\sqrt{2}} + \cdots + \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{2\sqrt{2}} \\
&= \frac{\alpha_1\left(\frac{\alpha_1^{2n-1}-1}{\alpha_1 - 1}\right) - \alpha_2\left(\frac{\alpha_2^{2n-1}-1}{\alpha_2 - 1}\right)}{2\sqrt{2}} \\
&= \frac{\alpha_1(\alpha_1^{2n-1}-1) + \alpha_2(\alpha_2^{2n-1}-1)}{4} \\
&= \frac{\alpha_1^{2n} + \alpha_2^{2n}}{4} - \frac{1}{2} \\
&= \frac{\alpha_1^{2n}(1-\alpha_2) - \alpha_2^{2n}(1-\alpha_1)}{4\sqrt{2}} - \frac{1}{2} \\
&= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} + \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} \\
&= B_n + b_n.
\end{aligned}$$

By virtue of Theorem 3.6.1, $b_n = R_n$ and the proof is complete. ■

It is also discussed in Chapter 3 that, if b is a cobalancing number with cobalancer r, then $b+r$ represents the side of the triangle in the figurate number representation of the pronic triangular number $b(b+1)$ as a triangle. The following theorem demonstrates the close association of this number $b+r$ with the partial sums of Pell sequence up to an even term.

4.2.5 Theorem. *The sum of first* $2n$ *Pell numbers is equal to the sum of* $(n+1)^{st}$ *cobalancing number and its cobalancer.*

Proof. Using the Binet formulas for P_n, B_n and b_n to the partial sum of first $2n$ Pell numbers, we get

$$\begin{aligned}
P_1+P_2+\cdots+P_{2n} &= \frac{\alpha_1-\alpha_2}{2\sqrt{2}}+\frac{\alpha_1^2-\alpha_2^2}{2\sqrt{2}}+\cdots+\frac{\alpha_1^{2n}-\alpha_2^{2n}}{2\sqrt{2}} \\
&= \frac{\alpha_1\left(\frac{\alpha_1^{2n}-1}{\alpha_1-1}\right)-\alpha_2\left(\frac{\alpha_2^{2n}-1}{\alpha_2-1}\right)}{2\sqrt{2}} \\
&= \frac{\alpha_1(\alpha_1^{2n}-1)+\alpha_2(\alpha_2^{2n}-1)}{4} \\
&= \frac{\alpha_1^{2n+1}+\alpha_2^{2n+1}}{4}-\frac{1}{2} \\
&= \frac{\alpha_1^{2n+1}(1-\alpha_2)-\alpha_2^{2n+1}(1-\alpha_1)}{4\sqrt{2}}-\frac{1}{2} \\
&= \frac{\alpha_1^{2n+1}-\alpha_2^{2n+1}}{4\sqrt{2}}-\frac{1}{2}+\frac{\alpha_1^{2n}-\alpha_2^{2n}}{4\sqrt{2}} \\
&= b_{n+1}+B_n.
\end{aligned}$$

By virtue of Theorem 3.6.1, $B_n=r_{n+1}$ and the proof is complete. ■

The last two theorems establish the links among sum of Pell numbers up to odd and even order with balancing and cobalancing numbers. The next two theorems provide close association of partial sums of odd and even ordered Pell numbers with balancing and cobalancing numbers respectively.

The following theorem establishes a relation between partial sums of odd terms of the Pell sequence and balancing numbers.

4.2.6 Theorem. *The sum of first* n *odd terms of the Pell sequence is equal to the* n^{th} *balancing number.*

Proof. Using the Binet formulas for P_n and B_n to the sum of first n odd terms of the Pell sequence, we get

$$P_1 + P_3 + \cdots + P_{2n-1} = \frac{\alpha_1 - \alpha_2}{2\sqrt{2}} + \frac{\alpha_1^3 - \alpha_2^3}{2\sqrt{2}} + \cdots + \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{2\sqrt{2}}$$

$$= \frac{\alpha_1\left(\frac{\alpha_1^{2n}-1}{\alpha_1^2-1}\right) - \alpha_2\left(\frac{\alpha_2^{2n}-1}{\alpha_2^2-1}\right)}{2\sqrt{2}}$$

$$= \frac{(\alpha_1^{2n}-1)-(\alpha_2^{2n}-1)}{4\sqrt{2}}$$

$$= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} = B_n. \qquad \blacksquare$$

The following theorem establishes the connection between partial sums of even terms of the Pell sequence and cobalancing numbers.

4.2.7 Theorem. *The sum of first* n *even terms of the Pell sequence is equal to the* $(n+1)^{st}$ *cobalancing number.*

Proof. Using the Binet formulas for P_n and B_n to the sum of first n even terms of the Pell sequence, we get

$$P_2 + P_4 + \cdots + P_{2n} = \frac{\alpha_1^2 - \alpha_2^2}{2\sqrt{2}} + \frac{\alpha_1^4 - \alpha_2^4}{2\sqrt{2}} + \cdots + \frac{\alpha_1^{2n} - \alpha_2^{2n}}{2\sqrt{2}}$$

$$= \frac{\alpha_1^2\left(\frac{\alpha_1^{2n}-1}{\alpha_1^2-1}\right) - \alpha_2^2\left(\frac{\alpha_2^{2n}-1}{\alpha_2^2-1}\right)}{2\sqrt{2}}$$

$$= \frac{\alpha_1(\alpha_1^{2n}-1)-\alpha_2(\alpha_2^{2n}-1)}{4\sqrt{2}}$$

$$= \frac{\alpha_1^{2n+1} - \alpha_2^{2n+1}}{4\sqrt{2}} - \frac{1}{2} = b_{n+1}. \qquad \blacksquare$$

The following theorem relates partial sums of odd terms of the associated Pell sequence to the sum of balancing numbers and their respective balancers.

4.2.8 Theorem. *The sum of first n odd terms of the associated Pell sequence is equal to the sum of n^{th} balancing number and its balancer.*

Proof. Using the Binet formulas for Q_n, B_n and b_n to the sum of first n odd ordered terms of the associated Pell sequence, we get

$$Q_1 + Q_3 + \cdots + Q_{2n-1} = \frac{\alpha_1+\alpha_2}{2} + \frac{\alpha_1^3+\alpha_2^3}{2} + \cdots + \frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2}$$

$$= \frac{\alpha_1\left(\frac{\alpha_1^{2n}-1}{\alpha_1^2-1}\right)+\alpha_2\left(\frac{\alpha_2^{2n}-1}{\alpha_2^2-1}\right)}{2}$$

$$= \frac{(\alpha_1^{2n}-1)+(\alpha_2^{2n}-1)}{4}$$

$$= \frac{\alpha_1^{2n}+\alpha_2^{2n}}{4} - \frac{1}{2}.$$

In the proof of Theorem 4.2.4, it has been shown that the last expression is equal to $B_n + R_n$. ■

Similarly, the following theorem links partial sums of even terms of the associated Pell sequence to the sum of cobalancing numbers and their respective cobalancers.

4.2.9 Theorem. *The sum of first n even ordered terms of the associated Pell sequence is equal to the sum of $(n+1)^{st}$ cobalancing number and its cobalancer.*

Proof. Using the Binet formulas for Q_n, B_n and b_n to the sum of first n even ordered terms of the associated Pell sequence, we get

$$Q_2 + Q_4 + \cdots + Q_{2n} = \frac{\alpha_1^2+\alpha_2^2}{2} + \frac{\alpha_1^4+\alpha_2^4}{2} + \cdots + \frac{\alpha_1^{2n}+\alpha_2^{2n}}{2}$$

$$= \frac{\alpha_1^2\left(\frac{\alpha_1^{2n}-1}{\alpha_1^2-1}\right) + \alpha_2^2\left(\frac{\alpha_2^{2n}-1}{\alpha_2^2-1}\right)}{2}$$

$$= \frac{\alpha_1(\alpha_1^{2n}-1) + \alpha_2(\alpha_2^{2n}-1)}{4}$$

$$= \frac{\alpha_1^{2n+1}+\alpha_2^{2n+1}}{4} - \frac{1}{2}.$$

In the proof of Theorem 4.2.5, it has been shown that the last expression is equal to $b_{n+1} + r_{n+1}$. ■

The partial sums of associated Pell sequence up to even and odd ordered terms are also related to balancing and cobalancing numbers. The following theorem links partial sums of associated Pell sequence up to odd ordered terms with balancing numbers.

4.2.10 Theorem. *The sum of first* $2n-1$ *associated Pell numbers is equal to twice the* n^{th} *balancing number decreased by one.*

Proof. Using the Binet formulas for Q_n and B_n to the sum of first $2n-1$ associated Pell numbers, we get

$$\begin{aligned} Q_1+Q_2+\cdots+Q_{2n-1} &= \frac{\alpha_1+\alpha_2}{2}+\frac{\alpha_1^2+\alpha_2^2}{2}+\cdots+\frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2} \\ &= \frac{\alpha_1\left(\frac{\alpha_1^{2n-1}-1}{\alpha_1-1}\right)+\alpha_2\left(\frac{\alpha_2^{2n-1}-1}{\alpha_2-1}\right)}{2} \\ &= \frac{\alpha_1(\alpha_1^{2n-1}-1)-\alpha_2(\alpha_2^{2n-1}-1)}{2\sqrt{2}} \\ &= \frac{\alpha_1^{2n}-\alpha_2^{2n}}{2\sqrt{2}}-1 \\ &= 2B_n-1. \end{aligned}$$

■

The following theorem links partial sums up to even ordered terms of the associated Pell sequence with cobalancing numbers.

4.2.11 Theorem. *The sum of first* $2n$ *associated Pell numbers is equal to the twice the* $(n+1)^{st}$ *cobalancing number.*

Proof. Using the Binet formulas for Q_n and b_n to the sum of first $2n$ associated Pell numbers, we get

$$\begin{aligned} Q_1+Q_2+\cdots+Q_{2n} &= \frac{\alpha_1+\alpha_2}{2}+\frac{\alpha_1^2+\alpha_2^2}{2}+\cdots+\frac{\alpha_1^{2n}+\alpha_2^{2n}}{2} \\ &= \frac{\alpha_1\left(\frac{\alpha_1^{2n}-1}{\alpha_1-1}\right)+\alpha_2\left(\frac{\alpha_2^{2n}-1}{\alpha_2-1}\right)}{2} \\ &= \frac{\alpha_1(\alpha_1^{2n}-1)-\alpha_2(\alpha_2^{2n}-1)}{2\sqrt{2}} \\ &= \frac{\alpha_1^{2n+1}-\alpha_2^{2n+1}}{2\sqrt{2}}-1 \\ &= 2b_{n+1}. \end{aligned}$$

■

The following theorem establishes a link between differences of Lucas-balancing numbers and cobalancing numbers.

4.2.12 Theorem. *The difference of n^{th} and $(n-1)^{st}$ Lucas-balancing numbers is equal to the difference of the $(n+1)^{st}$ and $(n-1)^{st}$ cobalancing numbers.*

Proof. By virtue of equation (11) of Chapter 2,

$$C_n = 8B_{n-1} + 3C_{n-1},$$

and similarly, by Theorem 2.4.3, we have

$$B_{n+1} = 3B_n + C_n.$$

Thus,

$$\begin{aligned} C_n - C_{n-1} &= 8B_{n-1} + 2C_{n-1} \\ &= 2[B_{n-1} + (C_{n-1} + 3B_{n-1})] \\ &= 2(B_{n-1} + B_n). \end{aligned}$$

Since

$$2(B_1 + B_2 + \cdots + B_{n-1}) = b_n$$

by Theorem 3.4.1, it follows that

$$2(B_{n-1} + B_n) = b_{n+1} - b_{n-1}.$$

This ends the proof. ■

The following corollary, which is contained in the above theorem, links differences of Lucas balancing numbers and sums of balancing numbers.

4.2.13 Corollary. *The difference of n^{th} and $(n-1)^{st}$ Lucas-balancing number is equal to twice the sum of n^{th} and $(n-1)^{st}$ balancing numbers.*

Theorem 4.2.12 provides a link between differences of Lucas-balancing numbers and cobalancing numbers. The following theorem provides a link between differences of Lucas-cobalancing and balancing numbers.

4.2.14 Theorem. *The difference of* n^{th} *and* $(n-1)^{st}$ *Lucas-cobalancing numbers is equal to the difference of the* n^{th} *and the* $(n-2)^{nd}$ *balancing numbers.*

Proof. From equation (5) of Chapter 3, we have

$$c_n = 8b_{n-1} + 3c_{n-1} + 4.$$

Applying this result, the recurrence relation

$$b_{n+1} = 3b_n + \sqrt{8b_n^2 + 8b_n + 1} + 1$$

of Chapter 3 and Theorem 3.6.1 to $c_n - c_{n-1}$, we get

$$\begin{aligned} c_n - c_{n-1} &= 8b_{n-1} + 2c_{n-1} + 4 \\ &= 2[b_{n-1} + (3b_{n-1} + c_{n-1} + 1) + 1] \\ &= 2(b_{n-1} + b_n + 1) \\ &= 2(R_{n-1} + R_n + 1) \\ &= (2R_{n-1} + 1) + (2R_n + 1). \end{aligned}$$

Since

$$R_n = \frac{-(2B_n+1)+\sqrt{8B_n^2+1}}{2},$$

we have

$$\begin{aligned} 2R_n + 1 &= -2B_n + \sqrt{8B_n^2 + 1} \\ &= -2B_n + C_n. \end{aligned}$$

Hence,

$$c_n - c_{n-1} = C_n + C_{n-1} - 2(B_n + B_{n-1}). \qquad \cdots(1)$$

Using the recurrence relations for B_n and C_n, we get

$$C_n + \sqrt{8}B_n = \alpha_1^{2n}$$

and

$$C_n - \sqrt{8}B_n = \alpha_2^{2n}.$$

Thus,

$$C_{n-1} - \sqrt{8}B_{n-1} = \alpha_2^{2(n-1)}. \qquad \cdots(2)$$

On the other hand,

$$\left(3+\sqrt{8}\right)\left(C_n-\sqrt{8}B_n\right)=(3C_n-8B_n)+\sqrt{8}(C_n-3B_n)$$
$$=\alpha_1^2\left(C_n-\sqrt{8}B_n\right)$$
$$=\alpha_1^2\alpha_2^{2n}=\alpha_2^{2(n-1)}. \qquad \cdots(3)$$

Using (2) and (3), we get

$$C_{n-1}-\sqrt{8}B_{n-1}=(3C_n-8B_n)+\sqrt{8}(C_n-3B_n). \qquad \cdots(4)$$

Comparison of rational and irrational parts from left hand and right hand sides of (4) yields

$$C_{n-1}=3C_n-8B_n, \qquad \cdots(5)$$

and

$$B_{n-1}=3B_n-C_n. \qquad \cdots(6)$$

Now, using (5) and (6), we find

$$B_{n-2}=3B_{n-1}-C_{n-1}$$
$$=3(3B_n-C_n)-(3C_n-8B_n)$$
$$=17B_n-6C_n. \qquad \cdots(7)$$

Inserting (5) and (6) into (1) and using (7), we get

$$c_n-c_{n-1}=6C_n-16B_n$$
$$=B_n-(17B_n-6C_n)$$
$$=B_n-B_{n-2}. \qquad \blacksquare$$

The following theorem provides a relation between sums of Lucas-balancing and Lucas-cobalancing numbers of same order with differences of squares of two Pell numbers.

4.2.15 Theorem. *The sum of n^{th} Lucas-balancing and n^{th} Lucas-cobalancing number is equal to the difference of squares of the $(n+1)^{st}$ and $(n-1)^{st}$ Pell numbers.*

Proof. Using the Binet formula for P_n, we get

$$P_{n+1}^2 - P_{n-1}^2 = \left(\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{2\sqrt{2}}\right)^2 - \left(\frac{\alpha_1^{n-1}-\alpha_2^{n-1}}{2\sqrt{2}}\right)^2$$

$$= \frac{\alpha_1^{2n+2}+\alpha_2^{2n+2}-\alpha_1^{2n-2}-\alpha_2^{2n-2}}{8}$$

$$= \frac{(\alpha_1^{2n}-\alpha_2^{2n})(\alpha_1^2-\alpha_2^2)}{8} = \frac{\alpha_1^{2n}-\alpha_2^{2n}}{\sqrt{2}}.$$

Observing that

$$1-\alpha_2 = -(1-\alpha_1) = \sqrt{2},$$

and using the Binet formulas for C_n and c_n, we get

$$\frac{\alpha_1^{2n}-\alpha_2^{2n}}{\sqrt{2}} = \frac{\alpha_1^{2n}(1-\alpha_2)+\alpha_2^{2n}(1-\alpha_1)}{2}$$

$$= \frac{\alpha_1^{2n}+\alpha_2^{2n}+\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2}$$

$$= \frac{\alpha_1^{2n}+\alpha_2^{2n}}{2} + \frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2}$$

$$= C_n + c_n. \qquad \blacksquare$$

The following theorem establishes relations between Lucas-cobalancing numbers and sums of two consecutive balancing numbers.

4.2.16 Theorem. *The n^{th} Lucas-cobalancing number is equal to the sum of $(n-1)^{st}$ and n^{th} balancing numbers.*

Proof. Using the Binet formulas for B_n and c_n, we find

$$B_{n-1} + B_n = \frac{\alpha_1^{2(n-1)}-\alpha_2^{2(n-1)}}{4\sqrt{2}} + \frac{\alpha_1^{2n}-\alpha_2^{2n}}{4\sqrt{2}}$$

$$= \frac{\alpha_1^{2n}(1+\alpha_2^2)-\alpha_2^{2n}(1+\alpha_1^2)}{4\sqrt{2}}$$

$$= \frac{\alpha_1^{2n}(-2\sqrt{2}\alpha_2)-\alpha_2^{2n}(2\sqrt{2}\alpha_1)}{4\sqrt{2}}$$

$$= \frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2} = c_n. \qquad \blacksquare$$

4.2.17 Remark. The following alternative forms are also available for c_n. By Theorem 2.4.3,

$$B_{n-1} = 3B_n - C_n,$$

and since $P_{2n} = 2B_n$ and $Q_{2n} = C_n$, using Theorem 4.2.3, we get

$$\begin{aligned} c_n &= 4B_n - C_n \\ &= 2P_{2n} - Q_{2n}. \end{aligned}$$

In some previous theorems, links among sums and differences of certain class of numbers are already discussed. The following theorem links the arithmetic means of Pell and associated Pell numbers with balancing and cobalancing numbers respectively.

4.2.18 Theorem. *The arithmetic mean of* $(2n-1)^{st}$ *Pell and associated Pell numbers is equal to the* n^{th} *balancing number and the arithmetic mean of* $2n^{th}$ *Pell and associated Pell numbers is ½ more than the* $(n+1)^{st}$ *cobalancing number.*

Proof. Using the Binet formulas for P_n, Q_n and B_n to the arithmetic mean of the n^{th} odd ordered Pell and associated Pell numbers, we get

$$\begin{aligned} \frac{P_{2n-1}+Q_{2n-1}}{2} &= \frac{1}{2}\left[\frac{\alpha_1^{2n-1}-\alpha_2^{2n-1}}{2\sqrt{2}} + \frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2}\right] \\ &= \frac{-\alpha_1^{2n}\alpha_2(1+\sqrt{2})+\alpha_2^{2n}\alpha_1(1-\sqrt{2})}{4\sqrt{2}} \\ &= \frac{-\alpha_1^{2n}\alpha_1\alpha_2+\alpha_2^{2n}\alpha_1\alpha_2}{4\sqrt{2}} \\ &= \frac{\alpha_1^{2n}-\alpha_2^{2n}}{4\sqrt{2}} = B_n. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{P_{2n}+Q_{2n}}{2} &= \frac{1}{2}\left[\frac{\alpha_1^{2n}-\alpha_2^{2n}}{2\sqrt{2}} + \frac{\alpha_1^{2n}+\alpha_2^{2n}}{2}\right] \\ &= \frac{\alpha_1^{2n}(1+\sqrt{2})-\alpha_2^{2n}(1-\sqrt{2})}{4\sqrt{2}} \\ &= \frac{\alpha_1^{2n+1}-\alpha_2^{2n+1}}{4\sqrt{2}} \\ &= \left[\frac{\alpha_1^{2(n+1)-1}-\alpha_2^{2(n+1)-1}}{4\sqrt{2}} - \frac{1}{2}\right] + \frac{1}{2} \\ &= b_{n+1} + \frac{1}{2}. \end{aligned}$$

■

In Chapters 2 and 3, we have observed that, n is a balancing number if and only if n^2 is a triangular number, and n is a cobalancing number if and only if $n(n+1)$ is a triangular number. It is worth mentioning here that if n^2 is a triangular number then $n+n=2n$ is an even ordered Pell number, and if $n(n+1)$ is a triangular number then $n+(n+1)=2n+1$ is an odd ordered Pell number. The following theorem demonstrates these assertions.

4.2.19 Theorem. *If n^2 is a triangular number (i.e. n is a balancing number) then $2n$ is an even ordered Pell number, and if $n(n+1)$ is a triangular number (i.e. n is a cobalancing number) then $2n+1$ is an odd ordered Pell number. Conversely, if P is an even ordered Pell number then $P^2/4$ is a square triangular number and if P is an odd ordered Pell number then $(P^2-1)/4$ is a pronic triangular number.*

Proof. If n^2 is a triangular number then n is a balancing number, say $n=B_k$ for some k. By virtue of Theorem 4.2.2,

$$2n = 2B_k = P_{2k}.$$

Conversely, for every even ordered Pell number $P_{2k}, P_{2k}/2$ is a balancing number by Theorem 4.2.2 and hence $P_{2k}^2/4$ is a square triangular number.

Further, using the Binet formulas for P_n and b_n, we get

$$\begin{aligned}\frac{P_{2k-1}-1}{2} &= \frac{1}{2}\left(\frac{\alpha_1^{2k-1}-\alpha_2^{2k-1}}{2\sqrt{2}}-1\right)\\ &= \frac{\alpha_1^{2k-1}-\alpha_2^{2k-1}}{4\sqrt{2}}-\frac{1}{2}=b_k. \qquad \cdots(8)\end{aligned}$$

It is known from Chapter 3 that n is a cobalancing number, if and only if $n(n+1)$ is a triangular number. Hence if $n(n+1)$ is a triangular number, then $n=b_k$ for some k, and using the Binet formula for b_n we find

$$\begin{aligned}2n+1 &= 2b_k+1\\ &= \frac{\alpha_1^{2k-1}-\alpha_2^{2k-1}}{2\sqrt{2}}=P_{2k-1}.\end{aligned}$$

Conversely, if P is an odd ordered Pell number, say $P = P_{2k-1}$ for some k, then by (8),

$$(P^2 - 1)/4 = \left(\frac{P-1}{2}\right)\left(\frac{P+1}{2}\right)$$
$$= b_k(b_k + 1),$$

which is a pronic triangular number. ■

The following theorem establishes associations of Pell and associated Pell numbers in the factorization of cobalancing numbers (balancers).

4.2.20 Theorem. *For $n = 1, 2, \cdots$, the $2n^{th}$ cobalancing number $(balancer)$ is equal to the product of $2n^{th}$ Pell number and the $(2n-1)^{st}$ associated Pell number, and the $(2n+1)^{th}$ cobalancing number is equal to the product of $2n^{th}$ Pell number and the $(2n+1)^{st}$ associated Pell number. More precisely, $b_{2n} = P_{2n}Q_{2n-1}$ and $b_{2n+1} = P_{2n}Q_{2n+1}$.*

Proof. Using Theorem 3.6.1 and the Binet forms of P_n, Q_n and b_n, we obtain

$$\begin{aligned} P_{2n}Q_{2n-1} &= \frac{\alpha_1^{2n}-\alpha_2^{2n}}{2\sqrt{2}} \cdot \frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2} \\ &= \frac{\alpha_1^{4n-1}-\alpha_2^{4n-1}-\alpha_1+\alpha_2}{4\sqrt{2}} \\ &= \frac{\alpha_1^{4n-1}-\alpha_2^{4n-1}}{4\sqrt{2}} - \frac{1}{2} = b_{2n}, \end{aligned}$$

and

$$\begin{aligned} P_{2n}Q_{2n+1} &= \frac{\alpha_1^{2n}-\alpha_2^{2n}}{2\sqrt{2}} \cdot \frac{\alpha_1^{2n+1}+\alpha_2^{2n+1}}{2} \\ &= \frac{\alpha_1^{4n+1}-\alpha_2^{4n+1}-\alpha_1+\alpha_2}{4\sqrt{2}} \\ &= \frac{\alpha_1^{2(2n+1)-1}-\alpha_2^{2(2n+1)-1}}{4\sqrt{2}} - \frac{1}{2} = b_{2n+1}. \end{aligned}$$ ■

Theorem 4.2.1 gives the basic relationship of balancing numbers with Pell and associated Pell numbers. The following theorem establishes the presence of Pell and associated Pell numbers as the greatest common divisors of balancing and cobalancing numbers (balancers).

4.2.21 Theorem. *The greatest common divisor of a balancing and a cobalancing number (balancer) of same order is either a Pell number, or an associated Pell number of the same order. More precisely,* $(B_{2n-1}, R_{2n-1}) = Q_{2n-1}$ *and* $(B_{2n}, R_{2n}) = P_{2n}$.

Proof. By virtue of Theorems 4.2.1 and 4.2.20, and using the fact that consecutive Pell numbers are relatively prime, we obtain

$$\begin{aligned}(B_{2n-1}, R_{2n-1}) &= (P_{2n-1}\, Q_{2n-1}, P_{2n-2}Q_{2n-1}) \\ &= Q_{2n-1}(P_{2n-1}, P_{2n-2}) = Q_{2n-1}.\end{aligned}$$

Further using Theorems 4.2.1 and 4.2.20, and the fact that consecutive associated Pell numbers are also relatively prime, we obtain

$$\begin{aligned}(B_{2n}, R_{2n}) &= (P_{2n}\, Q_{2n}, P_{2n}Q_{2n-1}) \\ &= P_{2n}(Q_{2n}, Q_{2n-1}) = P_{2n}.\end{aligned}$$ ■

The following theorem, which is similar to Theorem 4.2.21, establishes the presence of Pell and associated Pell numbers in the greatest common divisors of consecutive cobalancing numbers (balancers).

4.2.22 Theorem. *The greatest common divisor of two consecutive cobalancing numbers (balancers) is either twice of an odd ordered associated Pell number or is an even ordered Pell number. More precisely,* $(R_{2n-1}, R_{2n}) = 2Q_{2n-1}$ *and* $(R_{2n}, R_{2n+1}) = P_{2n}$.

Proof. Since consecutive Pell numbers are relatively primes, and the greatest common divisor of any two consecutive even ordered Pell numbers is 2, an use of Theorem 4.2.20 gives

$$\begin{aligned}(R_{2n-1}, R_{2n}) &= (P_{2n-2}\, Q_{2n-1}, P_{2n}Q_{2n-1}) \\ &= Q_{2n-1}(P_{2n-2}, P_{2n}) = 2Q_{2n-1},\end{aligned}$$

and

$$\begin{aligned}(R_{2n}, R_{2n+1}) &= (Q_{2n-1}\, P_{2n}, P_{2n}Q_{2n+1}) \\ &= P_{2n}(Q_{2n-1}, Q_{2n+1}) \\ &= P_{2n}(Q_{2n-1}, 2Q_{2n} + Q_{2n-1}) \\ &= P_{2n}(Q_{2n-1}, 2Q_{2n}) = P_{2n}.\end{aligned}$$ ■

4.3 SOME DIOPHANTINE EQUATIONS

In this section, we consider some simple Diophantine equations whose solutions are expressed as suitable combinations of balancing, cobalancing, Pell and associated Pell numbers.

The following theorem deals with the solution of a beautiful Diophantine equation, consists of finding two natural numbers such that, the sum of all natural numbers from the smaller number to the larger is equal to the product of these two numbers.

4.3.1 Theorem. *The solutions of the Diophantine equation* $x + (x+1) + (x+2) + \cdots + (x+y) = x(x+y)$ *are* $x = R_n + 1,\ y = B_n - R_n - 1,\ n = 1, 2, \cdots$.

Proof. Simplifying the equation

$$x + (x+1) + (x+2) + \cdots + (x+y) = x(x+y),$$

we get

$$\frac{y(y+1)}{2} = x(x-1),$$

showing that the pronic number $x(x-1)$ is also triangular. Hence, by virtue of the discussion in Section 3.1, $x-1$ must be a cobalancing number. Letting $x - 1 = b_n$ for some n, the last equation can be rewritten as

$$\frac{y(y+1)}{2} = b_n(b_n+1),$$

which gives

$$y = \frac{-1+\sqrt{8b_n^2+8b_n+1}}{2}$$

(the other root is not acceptable since y is positive). Now,

$$x + y = b_n + 1 + \frac{-1+\sqrt{8b_n^2+8b_n+1}}{2}$$

$$= \frac{3b_n+1+\sqrt{8b_n^2+8b_n+1}-b_n}{2}.$$

Since

$$3b_n + 1 + \sqrt{8b_n^2 + 8b_n + 1} = b_{n+1}$$

by Theorem 3.2.2, it follows that

$$x + y = \frac{b_{n+1} - b_n}{2}.$$

The right hand side of the last equation being equal to B_n, by virtue of Corollary 3.4.2, the totality of solutions are given by

$$x = B_n - b_n - 1, y = b_n + 1, n = 1, 2, \cdots.$$

Since for each n, $b_n = R_n$ by Theorem 3.6.2, the proof is complete. ■

We provide below an alternative proof of this theorem using Pell's equation:

The Diophantine equation

$$x + (x + 1) + \cdots + (x + y) = x(x + y)$$

is equivalent to

$$(2y + 1)^2 - 2(2x - 1)^2 = -1.$$

Setting $u = 2y + 1$ and $v = 2x - 1$, in the last equation, we get the Pell's equation

$$u^2 - 2v^2 = -1,$$

with both u and v odd. The fundamental solution of this equation is $u = 1, v = 1$. Hence, the totality of solutions is given by

$$u_n + \sqrt{2}v_n = \left(1 + \sqrt{2}\right)^n = \alpha_1^n ; n = 1, 2, \cdots.$$

Since this implies

$$u_n - \sqrt{2}v_n = \left(1 - \sqrt{2}\right)^n = \alpha_2^n;\ n = 1, 2, \cdots$$

we have

$$u_n = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2} = Q_n$$

and

$$v_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{2\sqrt{2}} = P_n.$$

Since both u_n and v_n are odd and P_n is odd only when n is odd, we must have

$$u_n = Q_{2n-1},\ v_n = P_{2n-1}; n = 1, 2, \cdots.$$

Thus,

$$2y + 1 = Q_{2n-1}, 2x - 1 = P_{2n-1}$$

implying that

$$y = (Q_{2n-1} - 1)/2, x = (P_{2n-1} + 1)/2.$$

Using the Binet formulas for P_n and R_n, it can be easily verified that

$$x = \frac{(P_{2n-1}+1)}{2}$$
$$= R_n + 1,$$

and using Theorem 4.2.18, we get

$$x + y = \frac{(P_{2n-1}+Q_{2n-1})}{2}$$
$$= B_n,$$

from which the conclusion of the theorem follows. ■

The following theorem, which resembles the previous theorem, deals with the solution of a Diophantine equation, which consists of finding two natural numbers such that the sum of all natural numbers from next to the smaller number to the larger number is equal to the product of the two numbers.

4.3.2 Theorem. *The solutions of the Diophantine equation* $(x + 1) + (x + 2) + \cdots + (x + y) = x(x + y)$ *are* $x = B_n, y = R_{n+1} - B_n$, $n = 1, 2, \cdots$.

Proof. The Diophantine equation

$$(x + 1) + (x + 2) + \cdots + (x + y) = x(x + y)$$

is equivalent to

$$\frac{y(y+1)}{2} = x^2.$$

This indicates that x^2 is a square triangular number, and hence x is a balancing number. Letting $x = B_n$ and solving the last equation for y we get

$$y = \frac{-1+\sqrt{8B_n^2+1}}{2}$$
$$= B_n + \frac{-(2B_n+1)+\sqrt{8B_n^2+1}}{2}$$
$$= B_n + R_n.$$

Thus, the totality of solutions are given by

$$x = B_n, y = B_n + R_n, n = 1, 2, \cdots.$$

Once again, we provide an alternative proof of this theorem using Pell's equation:

The Diophantine equation

$$(x + 1) + (x + 2) + \cdots + (x + y) = x(x + y)$$

is equivalent to

$$(2y + 1)^2 - 2(2x)^2 = 1.$$

Setting

$$u = 2y + 1, v = 2x,$$

we get the Pell's equation

$$u^2 - 2v^2 = 1,$$

with u and v both even. The fundamental solution of this equation is $u = 3$ and $v = 2$. Hence the totality of solutions may be given by

$$u_n + \sqrt{2}v_n = \left(3 + 2\sqrt{2}\right)^n, n = 1, 2, \cdots.$$

Since

$$u_n - \sqrt{2}v_n = \left(3 - 2\sqrt{2}\right)^n, n = 1, 2, \cdots,$$

using Theorems 3.2 and 3.3, we get

$$\begin{aligned} u_n &= \frac{\left(3+2\sqrt{2}\right)^n + \left(3-2\sqrt{2}\right)^n}{2} \\ &= \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2} = C_n = Q_{2n}, \end{aligned}$$

and

$$\begin{aligned} v_n &= \frac{\left(3+2\sqrt{2}\right)^n - \left(3-2\sqrt{2}\right)^n}{2\sqrt{2}} \\ &= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{2\sqrt{2}} \\ &= 2B_n = P_{2n}. \end{aligned}$$

We observe that Q_{2n} is always odd and P_{2n} is always even. Thus

$$2y + 1 = C_n, 2x = 2B_n$$

implying that

$$y = (C_n - 1)/2, x = B_n.$$

Now, by Corollary 3.6.4,

$$x + y = \frac{2B_n + C_n - 1}{2}$$

$$= \frac{-(2B_n + 1) + C_n}{2} + 2B_n$$

$$= R_n + 2B_n$$

$$= R_{n+1}. \qquad \blacksquare$$

The following theorem deals with the solution of a Diophantine equation consists of finding two natural numbers, the larger one being even, such that the sum of all natural from the smaller to the larger number is equal to the square of the smaller number.

4.3.3 Theorem. *The solutions of the Diophantine equation* $1 + 2 + \cdots + 2x = y^2$ *are* $x = {P_{2n}}^2 = 4{B_n}^2$ *and* $y = B_{2n}$, $n = 1, 2, \cdots$.

Proof. The Diophantine equation

$$1 + 2 + \cdots + 2x = y^2$$

is equivalent to

$$x(2x + 1) = y^2.$$

Since x and $2x + 1$ are relatively prime to each other, both x and $2x + 1$ must be squares. Letting

$$2x + 1 = (2l + 1)^2$$

we get

$$x = 4 \cdot \frac{l(l+1)}{2}.$$

Since x is a square, it follows that $l(l + 1)/2$ is a square triangular number. Hence,

$$x = 4B_n^2 = P_{2n}^2, \ n = 1, 2, \cdots,$$

and by virtue of Theorem 4.2.1

$$y = \sqrt{x(2x + 1)}$$

$$= \sqrt{4B_n^2(8B_n^2 + 1)}$$

$$= \sqrt{4B_n^2 C_n^2}$$

$$= 2B_n C_n = B_{2n} \qquad \blacksquare$$

The following theorem deals with the solution of a Diophantine equation, consists of finding two natural numbers, the larger one being odd, such that the sum of all natural numbers from the smaller to the larger number is equal to the square of the smaller number.

4.3.4 Theorem. *The solutions of the Diophantine equation* $1+2+\cdots+(2x-1)=y^2$ *are* $x=P_{2n-1}^2$ *and* $y=B_{2n-1}$, $n=1,2,\cdots$.

Proof. The Diophantine equation

$$1+2+\cdots+(2x-1)=y^2$$

is equivalent to

$$x(2x-1)=y^2.$$

Since x and $2x-1$ are relatively prime to each other, both x and $2x-1$ must be squares. Letting

$$2x-1=(2k+1)^2,$$

we get

$$x=2k^2+2k+1=k^2+(k+1)^2.$$

Taking $x=l^2$, the last equation takes the form

$$k^2+(k+1)^2=l^2.$$

By virtue of Theorem 3.7.1, the solutions of this Pythagorean equation are given by

$$k=b_n+r_n,$$

and

$$l=\sqrt{2k^2+2k+1}.$$

Using Binet formulas for B_n and b_n, and using Theorem 3.6.2, it can be easily seen that

$$l=P_{2n-1}.$$

Thus,

$$x=P_{2n-1}^2$$

and using Theorem 4.3.1, and the fact that

$$2P_{2n-1}^2-1=Q_{2n-1}^2,$$

we get,

$$\begin{aligned} y &= \sqrt{P_{2n-1}^2(2P_{2n-1}^2-1)} \\ &= P_{2n-1}Q_{2n-1}=B_{2n-1}. \end{aligned}$$ ■

In Theorems 4.3.3 and 4.3.4, if we keep the larger number unrestricted, we have the following theorem.

4.3.5 Theorem. *The solutions of the Diophantine equation* $1 + 2 + \cdots + x = y^2$ *are* $x = B_n + R_n$ *which is approximately equal to* Q_n^2, *and* $y = B_n$.

Proof. The Diophantine equation

$$1 + 2 + \cdots + x = y^2$$

is equivalent to

$$\frac{x(x+1)}{2} = y^2,$$

implying that y^2 is a square triangular number. Taking $y = B_n$, and using the Binet formulas for B_n, R_n and Q_n, it can be easily verified that

$$x = B_n + R_n = \begin{cases} Q_n^2, & \text{if } n \text{ is odd} \\ Q_n^2 - 1, & if \ \ n \text{ is even.} \end{cases} \qquad \blacksquare$$

In the following theorem, we consider finding two natural numbers such that the sum of natural numbers from the smaller to the larger number decreased by the smaller is equal to the square of the smaller number.

4.3.6 Theorem. *The solutions of the Diophantine equation* $1 + 2 + \cdots + (y - 1) + (y + 1) + \cdots + x = y^2$ *are* $x = b_n + r_n$ *and* $y = b_n, n = 1, 2, \cdots$.

Proof. The Diophantine equation

$$1 + 2 + \cdots + (y - 1) + (y + 1) + \cdots + x = y^2$$

is equivalent to

$$\frac{x(x+1)}{2} = y(y + 1).$$

This implies that $y(y + 1)$ is a pronic triangular number and hence, y is a cobalancing number. Taking $y = b_n$, and using Theorem 3.6.1, we get

$$x = \frac{-1+\sqrt{8b_n^2+8b_n+1}}{2} = b_n + r_n, \ n = 1, 2, \ \cdots. \qquad \blacksquare$$

Like Theorem 4.3.6, in the following theorem, we consider finding two natural numbers such that the larger one is even, and the sum of natural numbers up to the larger number decreased by the smaller is equal to the square of the smaller number.

4.3.7 Theorem. *The Diophantine equation* $1+2+\cdots+(y-1)+(y+1)+\cdots+2x=y^2$ *has no solution if x is odd. If x is even, the solutions are given by* $x=(b_{2n}+r_{2n})/2$ *and* $y=b_{2n}, n=1,2,\cdots$.

Proof. The Diophantine equation

$$1+2+\cdots+(y-1)+(y+1)+\cdots+2x=y^2$$

is equivalent to

$$x(2x+1)=y(y+1).$$

If x is odd, then the left hand side of the above equation is also odd, but the right hand side is even. Hence, in this case, no solution exists. If x is even, solving the above equation for x, we get

$$x=\frac{-1+\sqrt{8y^2+8y+1}}{4},$$

showing that $8y^2+8y+1$ is a perfect square. Hence y is a cobalancing number, say $y=b_n$. Since

$$r_n=\frac{-(2b_n+1)+\sqrt{8b_n^2+8b_n+1}}{2},$$

we have

$$x=(b_n+r_n)/2.$$

But b_n+r_n is even only when n is even. Hence the totality of solutions is given by

$$x=(b_{2n}+r_{2n})/2, y=b_{2n}, n=1,2,\cdots. \qquad \blacksquare$$

Like Theorem 4.3.7, in the following theorem, we consider finding two natural numbers such that the larger one is odd, and the sum of natural numbers up to the larger number decreased by the smaller is equal to the square of the smaller number.

4.3.8 Theorem. *The Diophantine equation* $1 + 2 + \cdots + (y-1) + (y+1) + \cdots + (2x-1) = y^2$ *has no solution if* x *is odd. If* x *is even, the solutions are given by* $x = (b_{2n-1} + r_{2n-1} + 1)/2$ *and* $y = b_{2n}$, $n = 1, 2, \cdots$.

Proof. The Diophantine equation

$$1 + 2 + \cdots + (y-1) + (y+1) + \cdots + (2x-1) = y^2$$

is equivalent to

$$x(2x-1) = y(y+1).$$

If x is odd, then $x(2x-1)$ is also odd, but $y(y+1)$ is even. Hence, in this case, no solution exists. If x is even, solving the above equation for x, we get

$$x = \frac{1+\sqrt{8y^2+8y+1}}{4},$$

showing that $8y^2 + 8y + 1$ is a perfect square. Hence, it follows y is a cobalancing number. Taking $y = b_n$, we get

$$x = (b_n + r_n + 1)/2.$$

But $b_n + r_n + 1$ is even only when n is odd. Thus, the totality of solutions in this case are given by

$$x = (b_{2n-1} + r_{2n-1} + 1)/2, y = b_{2n-1}, n = 1, 2, \cdots. \qquad \blacksquare$$

The Pythagorean equation $x^2 + (x+1)^2 = y^2$ has been studied in Chapters 2 and 3, wherein, the solutions are obtained in terms of balancing and cobalancing numbers. Let us call the equation $x^2 + y^2 = z^2 \pm 1$, the *almost Pythagorean equation*. In the following theorem, we consider the Diophantine equation $x^2 + (x+1)^2 = y^2 \pm 1$, which is a particular case of the almost Pythagorean equation.

4.3.9 Theorem. *The almost Pythagorean equation* $x^2 + (x+1)^2 = y^2 + 1$ *has the solutions* $x = B_n + b_n$ *and* $y = P_{2n}$ *,* $n = 1, 2, \cdots$ *, whereas, the equation* $x^2 + (x+1)^2 = y^2 - 1$ *has no solution.*

Proof. The Diophantine equation

$$x^2 + (x+1)^2 = y^2 + 1$$

is equivalent to

$$\frac{x(x+1)}{2} = \frac{y^2}{4},$$

showing that y is even, and $x(x+1)/2$ is a square triangular number. Hence $\sqrt{x(x+1)/2}$ is a balancing number. Writing

$$\frac{x(x+1)}{2} = B_n^2,$$

we find

$$y = 2B_n = P_{2n}.$$

Thus,

$$x = \frac{-1+\sqrt{8B_n^2+1}}{2}.$$

Since from the definition of balancing numbers and balancers

$$R_n = \frac{-(2B_n+1)+\sqrt{8B_n^2+1}}{2},$$

and by Theorem 3.6.2, $R_n = b_n$ for each n, it follows that

$$x = B_n + b_n.$$

The equation

$$x^2 + (x+1)^2 = y^2 - 1$$

is equivalent to

$$2(x^2 + x + 1) = y^2,$$

which indicates that y^2 is even and consequently, $x^2 + x + 1$ is also even. But this is a contradiction since $x^2 + x + 1$ is always odd. Hence, in this case, no solution exists. ■

The following theorem provides solutions of a Pythagorean equation in terms of balancing and Lucas-cobalancing numbers.

4.3.10 Theorem. *The Pythagorean equation* $x^2 + (x+2)^2 = y^2$ *has the solutions* $x = c_n - 1$ *and* $y = B_{2n+1}$, $n = 1, 2, \cdots$.

Proof. The Diophantine equation

$$x^2 + (x+2)^2 = y^2$$

is equivalent to

$$2(x^2 + 2x + 2) = y^2.$$

The latter indicates that y is even and hence $x^2 + 2x + 2$ is also even. Thus x is also even. Taking $x = 2u$ and $y = 2v$, the above equation reduces to

$$2u^2 + 2u + 1 = v^2,$$

which is the Pythagorean equation

$$u^2 + (u+1)^2 = v^2.$$

By virtue of Theorem 3.7.1, the solutions of this equation are given by

$$u = b_n + r_n, v = \sqrt{2u^2 + 2u + 1}.$$

Using the Binet formulas of b_n, r_n and P_n, it can be easily verified that

$$2(b_n + r_n)^2 + 2(b_n + r_n) + 1 = P_{2n+1}^2.$$

Since $r_n = B_{n-1}$ by Theorem 3.6.1, it follows that the solutions of

$$u^2 + (u+1)^2 = v^2$$

are given by

$$u = B_{n-1} + b_n, \quad v = P_{2n+1}, \quad n = 1, 2, \cdots.$$

Hence, the solutions of the Diophantine equation $x^2 + (x+2)^2 = y^2$ are given by

$$x = 2(B_{n-1} + b_n), \quad y = 2P_{2n+1} = B_{2n+1}, \quad n = 1, 2, \cdots.$$

Since

$$r_n = \frac{-(2b_n + 1) + c_n}{2},$$

it follows that

$$2(b_n + r_n) + 1 = c_n.$$

Hence, x can be alternatively given by $x = c_n - 1$. ■

Replacing x by $x - 1$ in the above theorem, we get the following interesting result.

4.3.11 Corollary. *The Pythagorean equation* $(x-1)^2+(x+1)^2=y^2$ *has the solutions* $x=c_n=Q_{2n-1}$ *and* $y=B_{2n+1},\ n=1,2,\ \cdots$.

In the following theorem, we consider a Pythagorean equation involving consecutive triangular numbers, whose solutions are obtained in terms of balancing numbers, associated Pell numbers and Lucas-cobalancing numbers.

4.3.12 Theorem. *The Pythagorean equation* $\left[\frac{x(x-1)}{2}\right]^2+\left[\frac{x(x+1)}{2}\right]^2=y^2$ *has the solutions* $x=Q_{2n-1}=c_n$ *and* $y=B_{2n-1}, n=1,2,\ \cdots$.

Proof. The Pythagorean equation

$$\left[\frac{x(x-1)}{2}\right]^2+\left[\frac{x(x+1)}{2}\right]^2=y^2$$

is equivalent to

$$\frac{x^2(x^2+1)}{2}=y^2, \qquad \cdots(9)$$

indicating that y^2 is a square triangular number. Hence, y is a balancing number, say $y=B_n$ for some n. Now solving (9) for x^2 and using the basic relationships between B_n and R_n, we get

$$x^2=\frac{-1+\sqrt{8B_n^2+1}}{2}=B_n+R_n.$$

In the proof of Theorem 4.3.5, we have shown that B_n+R_n is a perfect square only when n is odd and is equal to Q_n^2. Hence the totality of solutions of the Pythagorean equation

$$\left[\frac{x(x-1)}{2}\right]^2+\left[\frac{x(x+1)}{2}\right]^2=y^2$$

are given by $x=Q_{2n-1}$ and $y=B_{2n-1},\ n=1,2,\ \cdots$. Indeed, by virtue of Theorem 4.2.3, $Q_{2n-1}=c_n$. ■

4.4 PERFECT SQUARES RELATED TO BALANCING, COBALANCING AND RELATED NUMBERS

We have proved in Chapter 2 that if n is a balancing number then $8n^2+1$ is a perfect square. Further, in Chapter 3, we have seen that, if n is a cobalancing number then $8n^2+8n+1$ is a perfect square. These are not the only instances of perfect squares related to balancing and cobalancing numbers. There are ample examples of squares related to balancing, cobalancing and Lucas-balancing numbers.

The following theorem establishes relationships of balancing numbers and balancers with three different squares relating to Pell and associated Pell numbers.

4.4.1 Theorem. *Let B be any balancing number with balancer R. Then either $B+R$ is a perfect square or one less than a perfect square. In particular, if B is odd, then $B+R$ is a perfect square and if B is even, then both $B+R+1$ and $2(B+R)$ are perfect squares.*

Proof. It is obvious from the recurrence relation for B_n or Theorem 4.2.1 that B_{2n+1} is odd and B_{2n} is even for each n. By virtue of Theorem 4.2.1, 4.2.17 and the fact that

$$P_{n-1}+P_n=Q_n,$$

it follows that

$$\begin{aligned} B_{2n+1}+R_{2n+1} &= P_{2n+1}Q_{2n+1}+P_{2n}Q_{2n+1} \\ &= (P_{2n+1}+P_{2n})Q_{2n+1} \\ &= Q_{2n+1}^2. \end{aligned}$$

Again, since

$$Q_{n-1}+Q_n=2P_n,$$

we have

$$\begin{aligned} B_{2n}+R_{2n} &= P_{2n}Q_{2n}+P_{2n}Q_{2n-1} \\ &= P_{2n}(Q_{2n-1}+Q_{2n}) \\ &= 2P_{2n}^2. \end{aligned}$$

Thus,

$$2(B_{2n}+R_{2n}) = (2P_{2n})^2,$$

and

$$B_{2n}+R_{2n} + 1 = 2P_{2n}^2 + 1$$
$$= Q_{2n}^2. \qquad \blacksquare$$

The following theorem resembles Theorem 4.4.1 and establishes relationships of cobalancing numbers and cobalancers with squares of Pell and associated Pell numbers.

4.4.2 Theorem. *Let b be any cobalancing number with cobalancer r. Then either $b - r$ is a perfect square or one less than a perfect square. In particular, if r is odd then $b - r$ is a perfect square and if r is even, then both $b - r + 1$ and $2(b - r)$ are perfect squares.*

Proof. By virtue of Corollary 3.6.4, we have

$$R_n = R_{n-1} + 2B_{n-1}.$$

Further, in view of Theorem 3.6.2 and the last identity

$$b_n - r_n = R_n - B_{n-1}$$
$$= B_{n-1} + R_{n-1}.$$

The remaining part of the proof follows from Theorem 4.4.1. ■

As remarked at the beginning of this section, perfect squares are very closely related to Lucas-balancing numbers also. The following theorem establishes the close association of perfect squares with Lucas-balancing numbers.

4.4.3 Theorem. *For $n = 1,2,\cdots,$ the numbers $2(C_{2n} + 1)$, $C_{2n} - 1$, $C_{2n-1} + 1$, and $2(C_{2n-1} - 1)$ are perfect squares.*

Proof. By virtue of Corollary 2.4.9, we have

$$C_{2n} = C_n^2 + 8B_n^2$$
$$= 16B_n^2 + 1.$$

implying that

$$C_{2n} - 1 = 16B_n^2 = 4P_{2n}^2.$$

Further, by Theorem 4.2.3

$$2(C_{2n}+1) = 4C_n^2 = 4Q_{2n}^2.$$

Now using the Binet formulas for P_n, Q_n, C_n and c_n, we get

$$C_{2n-1}+1 = \frac{\alpha_1^{2(2n-1)}+\alpha_2^{2(2n-1)}}{2}+1$$

$$= 4\left[\frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2\sqrt{2}}\right]^2 = 4P_{2n-1}^2$$

and

$$(C_{2n-1}-1) = 4\left(\frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2}\right)^2$$

$$= 4c_n^2 = 4Q_{2n-1}^2.$$

This completes the proof. ■

CHAPTER 5

Some Important Properties of Balancing and Related Number Sequences

5.1 INTRODUCTION

In Chapter 2, we have seen that balancing numbers $n \in \mathbb{Z}^+$ and balancers $r \in \mathbb{Z}^+$ are solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

[also see 7]. Again in Chapter 3, cobalancing numbers $n \in \mathbb{Z}^+$ and cobalancers $r \in \mathbb{Z}^+$ are obtained as solutions of the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r).$$

It is well known that a natural number n is a balancing number if and only if n^2 is a triangular number, and thus, $8n^2 + 1$ is a perfect square and $\sqrt{8n^2 + 1}$ is called a Lucas-balancing number. Further, we have proved in Chapter 2 that the sequence $\{C_n\}_{n=1}^{n=\infty}$ of Lucas-balancing numbers satisfy the recurrence relation identical with that for balancing numbers.

In Chapter 3, it is seen that, if n is a cobalancing number, then $8n^2 + 8n + 1$ is a perfect square and the square root $\sqrt{8n^2 + 8n + 1}$ is called a Lucas-cobalancing number. The Lucas-cobalancing numbers $\{c_n\}_{n=1}^{n=\infty}$ also satisfy the recurrence relation for balancing numbers.

Behera and Panda [7] proved that

$$\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = 3 + \sqrt{8}.$$

Also, it can also be easily verified that

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 3 + \sqrt{8}.$$

Indeed, this also true for Lucas-balancing and Lucas-cobalancing numbers, that is,

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$$

$$= 3 + \sqrt{8}.$$

We write

$$\lambda_1 = 3 + \sqrt{8}$$

and

$$\lambda_2 = 3 - \sqrt{8}.$$

In what follows, we use the numbers λ_1 and λ_2 extensively to study some properties of balancing and related numbers.

5.2 SOME PROPERTIES OF THE NUMBERS λ_1 AND λ_2

In Chapters 2 and 3, we have seen that the numbers λ_1 and λ_2 occur in the Binet formulas for balancing and related numbers. In this section, we discuss some properties of the numbers λ_1 and λ_2 relating to the balancing numbers. We have already obtained the Binet formula for the sequence of balancing numbers in Chapter 2. Here, once again, we obtain the Binet formula for balancing numbers, but the method of derivation is different. We also obtain some bounds for the balancing numbers in terms of λ_1 and bounds for powers of λ_1 in terms of consecutive balancing numbers.

We first prove a theorem which demonstrates that like the integer sequences of Lucas-balancing and Lucas-cobalancing numbers, the irrational sequences $\{\lambda_1^n\}_{n=1}^{\infty}$ and $\{\lambda_2^n\}_{n=1}^{\infty}$ also satisfy the same recurrence relation as that of balancing numbers.

5.2.1 THEOREM. *For every natural number n, the following are true*:

(i) $\lambda_1^{n+1} = 6\lambda_1^n - \lambda_1^{n-1}$,

(ii) $\lambda_2^{n+1} = 6\lambda_2^n - \lambda_2^{n-1}$.

Proof. It is easy to verify that

$$\lambda_1^2 = 6\lambda_1 - 1,$$

and

$$\lambda_2^2 = 6\lambda_2 - 1$$

by direct calculation or by using the fact that λ_1 and λ_2 are solutions of the quadratic equation $x^2 - 6x + 1 = 0$ [7, p. 102]. Now multiplying both sides of these equations by λ_1^{n-1} and λ_2^{n-1} respectively, we get the desired results. ■

Note that, the above equations (i) and (ii) resemble the equation

$$B_{n+1} = 6B_n - B_{n-1},$$

which is the recurrence relation for the balancing numbers [7]. This resemblance turns out to be far from a coincidence. Besides this naive observation, a strong connection between the balancing numbers and the numbers λ_1 and λ_2 can be seen more explicitly in the next theorem, which gives the Binet formula for balancing number B_n in terms of λ_1 and λ_2.

5.2.2 Theorem. *For every natural number n,* $B_n = \frac{1}{2\sqrt{8}}(\lambda_1^n - \lambda_2^n)$.

Proof. This is nothing but the Binet formula of balancing numbers already discussed in Chapter 2. Here, we provide a proof of this theorem by induction.

Observe that for $n = 1$,

$$\frac{1}{2\sqrt{8}}(\lambda_1^1 - \lambda_2^1) = \frac{1}{2\sqrt{8}}\left[\left(3+\sqrt{8}\right) - \left(3-\sqrt{8}\right)\right]$$

$$= 1 = B_1.$$

Now, as an induction hypothesis, suppose that

$$B_i = \frac{1}{2\sqrt{8}}\left(\lambda_1^i - \lambda_2^i\right),$$

for all $i = 1, 2, \cdots, k$. Then,

$$\begin{aligned} B_{k+1} &= 6B_k - B_{k-1} \\ &= 6\left[\frac{1}{2\sqrt{8}}\left(\lambda_1^k - \lambda_2^k\right)\right] - \frac{1}{2\sqrt{8}}\left(\lambda_1^{k-1} - \lambda_2^{k-1}\right) \\ &= \frac{1}{2\sqrt{8}}\left[\left(6\lambda_1^k - \lambda_1^{k-1}\right) - \left(6\lambda_2^k - \lambda_2^{k-1}\right)\right] \\ &= \frac{1}{2\sqrt{8}}\left(\lambda_1^{k+1} - \lambda_2^{k+1}\right). \end{aligned}$$

Thus,

$$B_n = \frac{1}{2\sqrt{8}}\left(\lambda_1^n - \lambda_2^n\right),\ n = 1, 2, \cdots. \qquad \blacksquare$$

The number $\lambda_2 < 0.1716$. Thus for large n, λ_2^n is very close to 0. Hence, the role of λ_2^n in the Binet formula for B_n is negligible for large n, and thus, B_n is approximately equal to $\frac{1}{2\sqrt{8}}\lambda_1^n$.

5.2.3 Corollary. *For every natural number n, B_n is the integer closest to $\frac{1}{2\sqrt{8}}\lambda_1^n$. In particular, $B_n = \left\lceil\frac{1}{2\sqrt{8}}\lambda_1^n\right\rceil$, where $\lceil . \rceil$ denotes the ceiling function.*

Proof. By virtue of Theorem 5.2.2,

$$\begin{aligned} B_n &= \frac{1}{2\sqrt{8}}\left(\lambda_1^n - \lambda_2^n\right) \\ &= \frac{1}{2\sqrt{8}}\lambda_1^n - \frac{1}{2\sqrt{8}}\lambda_2^n. \end{aligned}$$

We would like to show that the second term is small in absolute value. Indeed, it suffices to show that this second term is less than $\frac{1}{4^{n+1}}$ in absolute value. Note that

$$\frac{1}{2\sqrt{8}} < \frac{1}{4},\ 0 < \lambda_2 < \frac{1}{4},$$

and hence,

$$\frac{1}{2\sqrt{8}}\lambda_2^n = \frac{1}{2\sqrt{8}}\lambda_2{}^n < \frac{1}{4^{n+1}}.$$

For each n, $\frac{1}{4^{n+1}} < 1$ and decreases exponentially with increasing n. Thus for large n, B_n is very close to $\frac{1}{2\sqrt{8}}\lambda_1^n$ and is equal to $\left\lceil\frac{1}{2\sqrt{8}}\lambda_1^n\right\rceil$. $\blacksquare$

It is clear from the above corollary that B_n grows somewhat in a exponential manner, in accordance with λ_1^n. The deviation of B_n from $\frac{1}{2\sqrt{8}}\lambda_1^n$ is measured by

$$\frac{1}{2\sqrt{8}}\lambda_2^n = \frac{1}{2\sqrt{8}}\lambda_2^n < \frac{1}{4^{n+1}},$$

which decreases fairly quickly as n grows, since it is exponential. We should note that B_n is slightly less than $\frac{1}{2\sqrt{8}}\lambda_1^n$.

Another interesting connection between balancing numbers and the number λ_1 is that the positive powers of λ_1 always lie between two consecutive balancing numbers. This, in turn, gives upper and lower bounds for the balancing numbers in terms of powers of λ_1.

The following theorem provides upper and lower bound for powers of λ_1 in terms of consecutive balancing numbers.

5.2.4 Theorem. *For each natural number* n, $B_n < \lambda_1^n < B_{n+1}$.

Proof. First, using the Binet formula for B_n, we get

$$B_n = \frac{\lambda_1^n}{2\sqrt{8}} - \frac{\lambda_2^n}{2\sqrt{8}}.$$

Since $\frac{\lambda_2^n}{2\sqrt{8}}$ is positive for each n, we have

$$B_n < \frac{\lambda_1^n}{2\sqrt{8}} < \lambda_1^n. \qquad \cdots (1)$$

Further, by virtue of Theorem 2.4.3,

$$B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1}.$$

Thus,

$$B_{n+1} > \left(3 + \sqrt{8}\right)B_n = \lambda_1 B_n.$$

Recursive iteration gives

$$B_{n+1} > \lambda_1^n B_1 = \lambda_1^n. \qquad \cdots (2)$$

Combining (1) and (2), we get the desired result. ■

Using inequalities (1) and (2), for each natural number n, a sharper bound for λ_1^n is

$$4\sqrt{2}B_n < \lambda_1^n < B_{n+1}.$$

The following corollary, which follows directly from Theorem 5.2.4, provides upper and lower bounds for balancing numbers.

5.2.5 Corollary. *For each natural number* n, $\lambda_1^{n-1} < B_n < \lambda_1^n$.

As usual, a sharper bound for B_n is

$$\lambda_1^{n-1} < B_n < \frac{\lambda_1^n}{4\sqrt{2}}.$$

Bounds for B_n can also be given by means of the powers of 5 and 6, which is demonstrated in the following theorem.

5.2.6 Theorem. *For every natural number* $n \geq 3$, $5^{n-1} < B_n < 6^{n-1}$.

Proof. For $n \geq 2$, the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}$$

gives

$$B_{n+1} < 6B_n.$$

Recursive iteration gives

$$B_{n+1} < 6^n B_1 = 6^n.$$

Thus, for $n \geq 3$

$$B_n < 6^{n-1}.$$

Now, it remains to show that, for $n \geq 3$, $B_n > 5^{n-1}$. We observe that,

$$\begin{aligned} B_{n+1} &= 6B_n - B_{n-1} \\ &= 5B_n + (B_n - B_{n-1}) \\ &> 5B_n\,, n \geq 1. \end{aligned}$$

Iterating recursively, we get

$$B_{n+1} > 5^n.$$

Thus,

$$B_n > 5^{n-1}$$

for $n \geq 2$ and the proof is complete. ∎

5.3 SOME MORE RESULTS ON BALANCING AND RELATED NUMBERS

In the previous section, we used the numbers λ_1 and λ_2 extensively in connection with the balancing numbers. In particular, we once again derived the Binet formula

$$B_n = \frac{1}{2\sqrt{8}}(\lambda_1^n - \lambda_2^n),\ n = 1, 2, \cdots$$

for the balancing numbers. In [7], it has been proved that these numbers appear as the roots of the auxiliary equation

$$x^2 - 6x + 1 = 0$$

while solving the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}$$

for balancing numbers. It is natural to wonder how and from where this numbers λ_1 and λ_2 arose. In this section, we will show that these numbers are eigen values of a 2×2 matrix whose powers generate the balancing numbers, and then investigate some more relationships of balancing and related numbers with this matrix. Throughout this section, $\mathbf{u}_0$ is a 2×1 matrix defined by $\mathbf{u}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{A}$ is a 2×2 matrix defined by $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 6 \end{bmatrix}$.

The following theorem gives a tool to generate balancing numbers of any order using the powers of the matrix $\mathbf{A}$.

5.3.1 Theorem. *The n^{th} power of the matrix $\mathbf{A}$ given by* $\mathbf{A}^n = \begin{bmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{bmatrix}$ *and consequently,* $B_n = \mathbf{u}_0^T \mathbf{A}^{n-1} \mathbf{u}_0$.

Proof. The proof is based on mathematical induction. For $n = 1$,

$$\mathbf{A}^n = \begin{bmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{bmatrix}$$

is obvious. Assume that

$$\mathbf{A}^n = \begin{bmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{bmatrix}$$

for $1 \le n \le k$. Then

$$\mathbf{A}^{k+1} = \mathbf{A}\mathbf{A}^{k} = \begin{bmatrix} 0 & 1 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} -B_{k-1} & B_k \\ -B_k & B_{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} -B_k & B_{k+1} \\ -6B_k + B_{k-1} & 6B_{k+1} - B_k \end{bmatrix}$$

$$= \begin{bmatrix} -B_k & B_{k+1} \\ -B_{k+1} & B_{k+2} \end{bmatrix},$$

showing that the assertion is true for $n = k + 1$. This completes the proof of the first part of the theorem.

The second part follows directly from the first part. ■

Not only the balancing numbers, but the cobalancing, Lucas-balancing and Lucas-cobalancing numbers can also be obtained as products of suitable matrices.

The following theorem gives a matrix product representation of the cobalancing numbers.

5.3.2 Theorem. L*et* $\mathbf{B}$ *be a* 2×2 *matrix defined by* $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}$. *Then* $\mathbf{A}^n\mathbf{B} = \begin{bmatrix} -B_n & 2R_{n+1} + 1 \\ -B_{n+1} & 2R_{n+2} + 1 \end{bmatrix}$. *Consequently,* $b_n = \frac{1}{2}(\mathbf{u}_0^T\mathbf{A}^{n-2}\mathbf{B}\mathbf{u}_0 - 1)$.

Proof. Using the value of $\mathbf{A}^n$ from the previous theorem, we get

$$\mathbf{A}^n\mathbf{B} = \begin{bmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -B_n & B_{n+1} - B_n \\ -B_{n+1} & B_{n+2} - B_{n+1} \end{bmatrix}. \qquad \cdots (3)$$

Further, by virtue of the nonlinear recurrence relation

$$B_{n-1} = 3B_n - \sqrt{8B_n^2 + 1}$$

from Theorem 2.4.3, and the basic formula

$$R_n = \frac{-(2B_n+1)+\sqrt{8B_n^2+1}}{2},$$

we get

$$B_n - B_{n-1} = -2B_n + \sqrt{8B_n^2 + 1}$$

$$= 2R_n + 1. \qquad \cdots (4)$$

Since $R_n = b_n$, by Theorem 3.6.2, (4) is equivalent to

$$B_n - B_{n-1} = 2b_n + 1. \qquad \cdots (5)$$

Applying (4) and (5) to (3), we get

$$\mathbf{A}^n\mathbf{B} = \begin{bmatrix} -B_n & 2b_{n+1}+1 \\ -B_{n+1} & 2b_{n+2}+1 \end{bmatrix}.$$

As usual, the second part follows from the first part. ■

The following theorem provides a method for expressing a Lucas-balancing number in terms of product of suitable matrices.

5.3.3 Theorem. *Let* $\mathbf{C}$ *be a* 2×2 *matrix defined by* $\mathbf{C} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. *Then* $\mathbf{A}^n\mathbf{C} = \begin{bmatrix} C_{n-1} & C_n \\ C_n & C_{n+1} \end{bmatrix}$, *and consequently,* $C_n = \mathbf{u}_0^T\mathbf{A}^{n-1}\mathbf{C}\mathbf{u}_0$.

Proof. Using the value of $\mathbf{A}^n$ from the Theorem 5.3.1, we get

$$\begin{aligned}\mathbf{A}^n\mathbf{C} &= \begin{bmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} B_n - 3B_{n-1} & 3B_n - B_{n-1} \\ B_{n+1} - 3B_n & 3B_{n+1} - B_n \end{bmatrix}.\end{aligned}$$

In view of the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1},$$

we have

$$B_{n+1} - 3B_n = 3B_n - B_{n-1},$$

and by virtue of the nonlinear recurrence relation

$$B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1}$$

we have

$$B_{n+1} - 3B_n = \sqrt{8B_n^2 + 1} = C_n.$$

Finally,

$$\begin{aligned}\mathbf{A}^n\mathbf{C} &= \begin{bmatrix} B_n - 3B_{n-1} & 3B_n - B_{n-1} \\ B_{n+1} - 3B_n & 3B_{n+1} - B_n \end{bmatrix} \\ &= \begin{bmatrix} C_{n-1} & C_n \\ C_n & C_{n+1} \end{bmatrix}.\end{aligned}$$

As usual, the second part follows from the first part. ■

Like the previous three theorems, the following theorem establishes a matrix product representation for the Lucas-cobalancing numbers.

5.3.4 Theorem. *Let* $\mathbf{D}$ *be a* 2×2 *matrix defined by* $\mathbf{D} = \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix}$*. Then* $\mathbf{A}^n\mathbf{D} = \begin{bmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{bmatrix}$, and *consequently,* $c_n = \mathbf{u}_0^T\mathbf{A}^{n-2}\mathbf{D}\mathbf{u}_0$.

Proof. Using the value of $\mathbf{A}^n$ from the Theorem 5.3.1, we get

$$\mathbf{A}^n\mathbf{D} = \begin{bmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} B_n + B_{n-1} & 7B_n - B_{n-1} \\ B_{n+1} + B_n & 7B_{n+1} - B_n \end{bmatrix}.$$

Further, using the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1},$$

we get

$$B_{n+1} + B_n = 7B_n - B_{n-1},$$

and by virtue of the Theorem 4.2.16, we have

$$B_{n-1} + B_n = c_n.$$

Finally,

$$\mathbf{A}^n\mathbf{D} = \begin{bmatrix} B_n + B_{n-1} & 7B_n - B_{n-1} \\ B_{n+1} + B_n & 7B_{n+1} - B_n \end{bmatrix}$$

$$= \begin{bmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{bmatrix}.$$

As usual, the second part follows from the first part. ■

In view of the relationships among balancing and related numbers with Pell and associated Pell numbers, it is clear from Theorems 5.3.1 – 5.3.4 that Pell and associated Pell numbers can also be represented in terms of the matrices $\mathbf{u}_0, \mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$. For example, in view of Theorems 4.2.2 and 5.3.1,

$$P_{2n} = 2\mathbf{u}_0^T\mathbf{A}^{n-1}\mathbf{u}_0.$$

Similarly, by virtue of Theorems 4.2.2 and 5.3.2,

$$P_{2n-1} = \mathbf{u}_0^T\mathbf{A}^{n-2}\mathbf{B}\mathbf{u}_0.$$

Further, by virtue of Theorems 4.2.3 and 5.3.3,

$$Q_{2n} = \mathbf{u}_0^T \mathbf{A}^{n-1} \mathbf{C} \mathbf{u}_0$$

and finally, by Theorems 4.2.3 and 5.3.4,

$$Q_{2n-1} = \mathbf{u}_0^T \mathbf{A}^{n-2} \mathbf{D} \mathbf{u}_0.$$

Using Theorem 5.3.1, all the balancing numbers can be derived from the powers of the matrix $\mathbf{A}$. We will conclude this section by giving an alternative proof of Theorem 5.2.2. In this proof, the numbers λ_1 and λ_2 arise naturally as eigen values of the matrix $\mathbf{A}$.

5.3.5 Alternative Proof of Theorem 5.2.2. The characteristic polynomial of the matrix $\mathbf{A}$ is given by

$$\begin{vmatrix} -\lambda & 1 \\ -1 & 6-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 1,$$

and the characteristic equation is

$$\lambda^2 - 6\lambda + 1 = 0.$$

Therefore, the eigen values of the matrix $\mathbf{A}$ are

$$\lambda = 3 \pm \sqrt{8}.$$

Thus, the eigen values of $\mathbf{A}$ are precisely λ_1 and λ_2, and it can be easily verified that the eigen vectors corresponding to these eigen values are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix},$$

and $\mathbf{u}_0$ as a linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$\mathbf{u}_0 = \frac{1}{2\sqrt{8}}(\mathbf{v}_1 - \mathbf{v}_2).$$

Now,

$$\mathbf{u}_0^T \mathbf{A}^{n-1}\, \mathbf{u}_0 = \frac{1}{2\sqrt{8}} \mathbf{u}_0^T \mathbf{A}^{n-1}(\mathbf{v}_1 - \mathbf{v}_2)$$

$$= \frac{1}{2\sqrt{8}} \mathbf{u}_0^T (\mathbf{A}^{n-1}\mathbf{v}_1 - \mathbf{A}^{n-1}\mathbf{v}_2).$$

Since λ_1 and λ_2 are eigen values of $\mathbf{A}$ with eigen vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively, the eigen values of $\mathbf{A}^{n-1}$ are also λ_1^{n-1} and λ_2^{n-1} with same eigen vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Hence

$$\mathbf{A}^{n-1}\mathbf{v}_1 = \lambda_1^{n-1}\mathbf{v}_1$$

and

$$\mathbf{A}^{n-1}\mathbf{v}_2 = \lambda_2^{n-1}\mathbf{v}_2.$$

Now,

$$\mathbf{u}_0^T\mathbf{A}^{n-1}\mathbf{u}_0 = \frac{1}{2\sqrt{8}}[\lambda_1^{n-1}\mathbf{u}_0^T\mathbf{v}_1 - \lambda_2^{n-1}\mathbf{u}_0^T\mathbf{v}_2]$$

$$= \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}}.$$

By virtue of Theorem 5.3.1,

$$\mathbf{u}_0^T\mathbf{A}^{n-1}\mathbf{u}_0 = B_n.$$

Hence,

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}},$$

which is nothing but the Binet formula for balancing numbers. ■

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